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New upper and lower bounds for the upper incomplete gamma function

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ABSTRACT

Some new upper and lower bounds for the upper incomplete gamma function $\Gamma(a,x)$ are presented. Some of the bounds are given for all real $x > 0$ and some are for only certain combinations of a and x. A number of different methods are used to obtain these new bounds. In particular, rational function and perturbations of rational function bounds are presented. Some numerical comparisons are made with previously proposed bounds.

1. Preliminary results and bounds

Let x and a be real numbers with $x > 0$. Then the upper incomplete gamma function is given by

$$\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt. \tag{1}$$

The lower incomplete gamma function is

$$\gamma(a,x) = \int_0^x t^{a-1}e^{-t}dt. \tag{2}$$

Clearly, $\Gamma(a,x) + \gamma(a,x) = \Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt$, the gamma function. We shall be primarily concerned with new upper and lower bounds on the upper incomplete gamma function (referred to from now on as just the incomplete gamma function). However, any upper (lower) bounds on $\Gamma(a,x)$ will give us a lower (upper) bound on $\gamma(a,x)$ upon subtraction from $\Gamma(a)$ if desired.

Before presenting the new bounds, we present some needed results and some previously proposed bounds for $\Gamma(a,x)$, especially these bounds given in [1]. Let

$$b_a = \begin{cases} \Gamma(a+1)^{1/(a-1)} & if \quad a \in (-1,\infty)\backslash\{1\}, \\ e^{1-\gamma} & if \quad a = 1, \end{cases}$$

where $\gamma = 0.577\cdots$ is the Euler's constant. Let

$$G_a(x) = \begin{cases} x^{-2}e^{-x} & if \quad a = -1, \\ \dfrac{(x+b_a)^a - x^a}{ab_a}e^{-x} & if \quad a \in (-1,\infty)\backslash\{0\}, \\ e^{-x}\ln\left(\frac{x+1}{x}\right) & if \quad a = 0. \end{cases}$$

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Let

$$g_a(x) = \left(\frac{(x+2)^a - x^a - 2^a}{2a} + \Gamma(a) \right) e^{-x}, \quad a > 0. \tag{3}$$

Many of the bounds given in [1] are based on $G_a(x)$ and $g_a(x)$ or their ‘forward’ or ‘backward’ shift variants. For any bound $B_a(x)$ on $\Gamma(a,x)$, we can obtain new bounds on $\Gamma(a,x)$ using the forward $k-$ shift:

$$B_{a;k}(x) = x^{a-1} e^{-x} \sum_{j=0}^{k-1} (a-1)_j x^{-j} + (a-1)_k B_{a-k}(x), \tag{4}$$

where $(a-1)_j = \prod_{i=0}^{j-1}(a-1-i)$ is the jth falling factorial of $a-1$. Clearly, a bound on $\Gamma(a,x)$ can be obtained using a bound on $\Gamma(a-k,x)$. The bound type may change if $(a-1)_k < 0$, however. See [1], p. 4–5 for details. Thus, in fact, it suffices to obtain bounds only for $0 < a < 1$, if the forward shift relation (4) is used. The bounds given in [1] are exact since $B_a(x) \underset{x\downarrow 0}{\sim} \Gamma(a,x)$ and $B_a(x) \underset{x\to\infty}{\sim} \Gamma(a,x)$ for most bound choices $B_a(x)$ constructed in [1]. This will not be the case for many of the new bounds presented in this paper. However, the rational function bounds and their perturbed versions will be exact in one way in that $B_a(x) \underset{x\to\infty}{\sim} \Gamma(a,x)$ for these new bounds. Moreover, for ‘intermediate’ values of $x > 0$, the new bounds can be improvements on some of the bounds given in [1] for certain choices of a and x.

For other previously proposed bounds on $\Gamma(a,x)$ and other works relevant to this paper, see the research works: [2–7], and [8].

Next, we present the bounds of Pinelis [1] most relevant to the results to be presented later in Sections 2–3.

Theorem 1.1 (*Proposition 2.7 of Pinelis [1]*)**.** *Let $a \geq 1$ be real. Then*

$$\Gamma(a,x) \geq x^{a-1} e^{-x}, \tag{5}$$

for $x > 0$ and

$$\Gamma(a,x) \leq \frac{x^{a-1} e^{-x}}{1 - \left(\frac{a-1}{x}\right)}, \quad \text{for} \quad x > a-1. \tag{6}$$

Theorem 1.2 (*Proposition 2.8 of Pinelis [1]*)**.** *Let $a < 1$ be real. Let*

$$g^{lo}_{a;2}(x) = x^a e^{-x} \left(\frac{x+3-a}{x^2 + (4-2a)x + (a-1)(a-2)} \right). \tag{7}$$

Then

$$\Gamma(a,x) > g^{lo}_{a;2}(x). \tag{8}$$

Theorem 1.3 (*Theorem 1.2 of Pinelis [1]*)**.** *Let $a < -1$ be real. Then for $x > 0$,*

$$g^{lo}_a(x) < \Gamma(a,x) < g^{up}_a(x), \tag{9}$$

where

$$g^{lo}_a(x) = x^a e^{-x} \left(\frac{x-a-1}{(x-a)^2 + a} \right), \tag{10}$$

and

$$g^{up}_a(x) = x^a e^{-x} \left(\frac{1}{x-a} \right). \tag{11}$$

Theorem 1.4 (*Theorem 1.1 of Pinelis [1]*)**.** *Let $a \geq -1$ be real. Then for $x > 0$,*

$$\begin{array}{ll}
\Gamma(a,x) < G_a(x) & \text{if} \quad -1 \leqslant a < 1, \\
g_a(x) = G_a(x) = \Gamma(a,x) = e^{-x} & \text{if} \quad a = 1, \\
g_a(x) < G_a(x) < \Gamma(a,x) & \text{if} \quad 1 < a < 2, \\
g_a(x) = G_a(x) = \Gamma(a,x) = e^{-x}(1+x) & \text{if} \quad a = 2, \\
\Gamma(a,x) < g_a(x) < G_a(x) & \text{if} \quad 2 < a < 3, \\
\Gamma(a,x) = g_a(x) = e^{-x}\left(2 + 2x + x^2\right) < G_a(x) & \text{if} \quad a = 3, \\
g_a(x) < \Gamma(a,x) < G_a(x) & \text{if} \quad a > 3.
\end{array} \tag{12}$$

Theorem 1.5 (*Proposition 2.11 of Pinelis [1]*)**.** *Let $a < 0$ be real. Let*

$$g^{lo}_{a;1}(x) = x^a e^{-x} \left(\frac{1-a+x}{(x-a)^2 - a + 2x} \right). \tag{13}$$

Then

$$\Gamma(a,x) < g^{lo}_{a;1}(x) < g^{up}_a(x), \tag{14}$$

where $g^{up}_a(x)$ is given in (11).

Theorem 1.6 (*Proposition 2.10 of Pinelis [1]*). *Let* $1 < a < 3$. *Let*

$$G_{a;1}(x) = x^{a-1}e^{-x} + (a-1)G_{a-1}(x). \tag{15}$$

Then

$$\begin{aligned} &\Gamma(a,x) < G_{a;1}(x) \quad \text{if} \quad 1 < a < 2, \\ &\Gamma(a,x) = G_{a;1}(x) \quad \text{if} \quad a = 2, \\ &\Gamma(a,x) > G_{a;1}(x) \quad \text{if} \quad 2 < a < 3. \end{aligned} \tag{16}$$

Theorem 1.7 (*Proposition 2.9 of Pinelis [1]*). *Let*

$$G_{a;-1}(x) = \frac{1}{a}\left(G_{a+1}(x) - x^a e^{-x}\right). \tag{17}$$

Then for $x > 0$*,*

$$\Gamma(a,x) > G_{a;-1}(x), \tag{18}$$

As discussed earlier, many new bounds can be given using the forward or backward $k-$shift. For example, using (17), the bound $G_{a;6}(x)$ can be found using $B_a(x) = G_a(x)$ with the forward $k = 6$ shift to get

$$G_{a;6}(x) = x^{a-1}e^{-x}\sum_{j=0}^{5}(a-1)_j x^{-j} + (a-1)_k G_{a-6}(x), \tag{19}$$

which is a bound used in Figure 1 of Pinelis [1] in the numerical graphical comparisons of signed relative errors of bounds. Now we are ready to give some new bounds for the incomplete gamma function.

2. Main results

Theorem 2.1. *For all real* a *and* $x > 0$*, let*

$$H(a,x) = \frac{\Gamma(a,x)}{x^{a-1}e^{-x}} = \frac{\int_x^\infty t^{a-1}e^{-t}dt}{x^{a-1}e^{-x}}. \tag{20}$$

Then $H(a,x)$ *is a log convex function of* a *for* $x > 0$.

Proof. From the definition of $H(a,x)$ given in (20) a change of variable gives

$$H(a,x) = \int_0^\infty \left(1+\frac{u}{x}\right)^{a-1} e^{-u}du, \tag{21}$$

Differentiating (21) with respect to a twice gives

$$\begin{aligned} \Delta &= H(a,x)H''(a,x) - \left(H'(a,x)\right)^2 \\ &= \left(\int_0^\infty \left(1+\frac{u_1}{x}\right)^{a-1} e^{-u_1}du_1\right)\left(\int_0^\infty \left(1+\frac{u_2}{x}\right)^{a-1}\right)\left(\log\left(1+\frac{u_2}{x}\right)\right)^2 e^{-u_2}du_2 \\ &- \left(\int_0^\infty \left(1+\frac{u_1}{x}\right)^{a-1} \cdot \log\left(1+\frac{u_1}{x}\right)e^{-u_1}du_1\right)\cdot\left(\int_0^\infty \left(1+\frac{u_2}{x}\right)^{a-1}\log\left(1+\frac{u_2}{x}\right)e^{-u_2}du_2\right), \end{aligned} \tag{22}$$

After some algebra and combining integrands, we get

$$\Delta = \int_0^\infty \int_0^\infty \frac{1}{2}(c_1-c_2)^2 e^{-u_1}e^{-u_2}\left(1+\frac{u_1}{x}\right)^{a-1}\left(1+\frac{u_2}{x}\right)^{a-1} du_1 du_2, \tag{23}$$

where $c_1 = \log\left(1+\frac{u_1}{x}\right)$ and $c_2 = \log\left(1+\frac{u_2}{x}\right)$. Clearly, (23) is nonnegative which gives $\frac{H'(a,x)}{H(a,x)}$ is increasing in a which is equivalent to $H(a,x)$ is log convex function of a for all $x > 0$. This completes the proof. ■

Next, we give an interpolatory type of upper bound on $\Gamma(a,x)$ using Theorem 2.1.

Theorem 2.2. *Let* $a \geq 1$ *and* $x > 0$. *Let* $m = \lfloor a \rfloor$ *be the floor function value of* a. *Then*

$$\Gamma(a,x) \leq t^{a-1}e^{-t} \cdot [H(m)]^{1-(a-m)} \cdot [H(m+1)]^{a-m} \cdot e^{[1-(a-m)]y_m + (a-m)y_{m+1}} \tag{24}$$

where

$$H(m) = \sum_{j=0}^{m-1} \frac{(m-1)!}{(m-1-j)!}\left(\frac{1}{x}\right)^j = H(m,x), \quad \textit{and} \quad y_m = \log H(m), \tag{25}$$

Proof. From the definition of $H(a,x)$ given in (20) with $a = m$, we get, using binomial expansions:

$$H(m+1) = \int_0^\infty \left(1+\frac{u}{x}\right)^m e^{-u}du = \sum_{j=0}^{m} \frac{m!}{(m-j)!}\left(\frac{1}{x}\right)^j. \tag{26}$$

Since $\log H(a,x)$ is concave in a, a linear interpolation in a will produce an overestimate of $\log H(a,x)$ when interpolating at the points $(m, \log H(m,x))$ and $(m+1, \log H(m+1,x))$. Thus for $m \le a \le m+1$,

$$\begin{aligned}\log H(a,x) &\le y_m + (y_{m+1} - y_m)(a-m) \\ &= [1-(a-m)]y_m + (a-m)y_{m+1}.\end{aligned} \tag{27}$$

Exponentiation of (27) gives,

$$H(a,x) \le e^{[1-(a-m)]y_m+(a-m)y_{m+1}}\left(H(m)\right)^{1-(a-m)}\left(H(m+1)\right)^{a-m}, \tag{28}$$

From the definition of $H(a,x)$, (24) follows immediately. This completes the proof. ■

Theorem 2.3 (*Theorem 4.3 of Fagiuoli and Pellerey [9]*). *Suppose U is a random variable having a new better than used in expectation (NBUE) distribution. Let $h(u)$ be a convex function of u with $h(0) \le 0$. Then*

$$E\left[h(U)\right] \le E(U)\cdot E(h'(U)). \tag{29}$$

From Theorem 2.3 above, we can prove the following result.

Theorem 2.4. *Suppose $a < 1$ and $x > 0$. Then*

$$\Gamma(a,x) \le \left(\frac{x+1}{x+2-a}\right)\cdot\left(x^{a-1}e^{-x}\right). \tag{30}$$

Proof. From the definition of $H(a,x)$,

$$H(a,x) = \left(\frac{1}{x}\right)^{a-1}\int_0^\infty \frac{1}{(u+x)^{1-a}}e^{-u}du. \tag{31}$$

Let

$$h(u) = \frac{1}{(u+x)^{1-a}} - \frac{1}{x^{1-a}}. \tag{32}$$

Then $h(0) = 0$. Also, $h(u)$ is convex function of $u \ge 0$ for all $x > 0$. In addition, the function

$$f_1(u) = (u+x)h'(u) = \frac{a-1}{(u+x)^{1-a}}, \tag{33}$$

is increasing in u for $x > 0$. From (31), we can write

$$H(a,x) = E\left[\frac{1}{(U+x)^{1-a}}\right]\cdot\left(\frac{1}{x}\right)^{a-1}, \tag{34}$$

where U has an exponential distribution, which is an NBUE distribution with mean 1 and probability density function $e^{-u}, u \ge 0$. Thus, Theorem 2.3 gives

$$\begin{aligned}E\left[h(U)\right] &= E\left[\frac{1}{(U+x)^{1-a}} - \frac{1}{x^{1-a}}\right] \le E(U)\cdot E(h'(U)) \\ &= E(h'(U)), \quad \text{since} \quad E(U) = 1.\end{aligned} \tag{35}$$

Since $f_1(u) = (u+x)h'(u)$ is increasing in u and $\dfrac{1}{u+x}$ is decreasing in u, standard covariance inequalities give

$$\begin{aligned}E[h(U)] \le E(h'(U)) &= E\left[(U+x)h'(U)\cdot\left(\frac{1}{U+x}\right)\right] \\ &\le E[(U+x)h'(U)]\cdot E\left[\frac{1}{U+x}\right].\end{aligned} \tag{36}$$

Since $h'(u) = \dfrac{a-1}{(u+x)^{2-a}}$, we get

$$E\left(\frac{1}{(U+x)^{1-a}}\right) \le \left(\frac{1}{x}\right)^{1-a}(a-1)E\left[\frac{1}{(U+x)^{1-a}}\right]\cdot E\left[\frac{1}{U+x}\right]. \tag{37}$$

Solving for $E\left[\dfrac{1}{(U+x)^{1-a}}\right]$ in (37), we get

$$E\left[\frac{1}{(U+x)^{1-a}}\right] \le \frac{\left(\frac{1}{x}\right)^{1-a}}{1+\frac{1-a}{1+x}}. \tag{38}$$

From (34), and (38), we get

$$H(a,x) = \left(\frac{1}{x}\right)^{1-a} E\left[\frac{1}{(U+x)^{1-a}}\right] \le \frac{1}{1+\frac{1-a}{1+x}} = \frac{x+1}{x+2-a}. \tag{39}$$

From the definition of $H(a,x)$, we get

$$\Gamma(a,x) \le \left(\frac{x+1}{x+2-a}\right) \cdot \left(x^{a-1}e^{-x}\right), \quad \text{the desired result.} \tag{40}$$

This completes the proof. ■

Recall that Theorem 1.1 (Proposition 2.7 of Pinelis [1]) stated that

$$\Gamma(a,x) \le \frac{x^{a-1}e^{-x}}{(1-(a-1)/x)}, \quad \text{for all real } a \ge 1 \text{ and all real } x > a-1. \tag{41}$$

Let us re-derive this in another way which leads to some new bounds for $\Gamma(a,x)$. In particular, we shall get an extension of Theorem 1.5 (Proposition 2.11 of Pinelis [1]) which states that

$$\Gamma(a,x) > g^{lo}_{a;1}(x) = \frac{x^a e^{-x}(1-a+x)}{(x-a)^2-a+2x}, \tag{42}$$

for $a \ge 2$ and $x > a-1$. Note that (42) has the inequality sign going the other way since $\Gamma(a,x) < g^{lo}_{a;1}(x)$ holds instead for $a < 0$ and $x > 0$. Let

$$G(x) = H(a,x) = \frac{\int_x^\infty t^{a-1}e^{-t}dt}{x^{a-1}e^{-x}}, \tag{43}$$

emphasizing the dependence of $H(a,x)$ on x instead. Then computing $G'(x)$ and $G''(x)$ the first two derivatives of $G(x)$, we get

$$G'(x) = \left(1-\left(\frac{a-1}{x}\right)\right)G(x)-1. \tag{44}$$

But from (43), we also have

$$G(x) = \int_0^\infty \left(1+\frac{u}{x}\right)^{a-1} e^{-u}, \tag{45}$$

which gives

$$G'(x) = \int_0^\infty (a-1)\left(1+\frac{u}{x}\right)^{a-1}\left(\frac{-u}{x^2}\right)e^{-u}du < 0. \tag{46}$$

From (46), we get, after using (44), that for $a \ge 1$,

$$\left(1-\left(\frac{a-1}{x}\right)\right)G(x)-1 < 0, \quad \text{if} \quad x > a-1 \tag{47}$$

which gives for $a \ge 1$ that

$$G(x) < \frac{1}{1-\left(\frac{a-1}{x}\right)}, \quad x > a-1 \tag{48}$$

and also that

$$\Gamma(a,x) < \frac{x^{a-1}e^{-x}}{1-\left(\frac{a-1}{x}\right)}, \quad x > a-1. \tag{49}$$

which is part of Proposition 2.7 of Pinelis [1]. Differentiating once more, we get

$$G''(x) = \int_0^\infty \left[(a-1)\left(1+\frac{u}{x}\right)^{a-2}\cdot\frac{2u}{x^3} + \left(\frac{-u}{x^2}\right)(a-1)(a-2)\left(1+\frac{u}{x}\right)^{a-3}\right]e^{-u}du \ge 0, \quad \text{if } a \ge 2. \tag{50}$$

Also, from (50), we get

$$G''(x) = \left[\left(1-\left(\frac{a-1}{x}\right)\right)^2 + \left(\frac{a-1}{x^2}\right)\right]\cdot G(x) - \left(1-\left(\frac{a-1}{x}\right)\right). \tag{51}$$

Thus, for $x > a-1$, we get

$$G(x) > \frac{1-\left(\frac{a-1}{x}\right)}{\left(1-\left(\frac{a-1}{x}\right)\right)^2 + \left(\frac{a-1}{x^2}\right)}. \tag{52}$$

From the definition of $H(a,x)$, we get

$$\Gamma(a,x) > x^{a-1}e^{-x}\left(\frac{1-\left(\frac{a-1}{x}\right)}{\left(1-\left(\frac{a-1}{x}\right)\right)^2+\left(\frac{a-1}{x^2}\right)}\right) = \frac{x^a e^{-x}(1-a+x)}{(x-a)^2-a+2x} = g^{lo}_{a;1}(x). \quad (53)$$

Thus, Theorem 2.5 below holds.

Theorem 2.5. *Suppose $a \geq 2$ and $x > a-1$. Then*

$$\Gamma(a,x) > \frac{x^a e^{-x}(1-a+x)}{(x-a)^2-a+2x} = g^{lo}_{a;1}(x). \quad (54)$$

Theorem 2.6. *Let a be real and $x > a$ be real. Let*

$$M_1(a,x) = x^{a-1}e^{-x}\left(\frac{x-\left(1+\frac{2}{x}\right)^{a-1}}{x-a}\right). \quad (55)$$

Then

$$\Gamma(a,x) \leq M_1(a,x) \;\; \text{if} \;\; a \geq 2, \quad (56)$$

$$\Gamma(a,x) \geq M_1(a,x) \;\; \text{if} \;\; 1 \leq a < 2, \quad (57)$$

and

$$\Gamma(a,x) \leq M_1(a,x) \;\; \text{if} \;\; a < 1. \quad (58)$$

Proof. Since

$$G(x) = \int_0^\infty \left(1+\frac{u}{x}\right)^{a-1} e^{-u}du, \quad (59)$$

$$R(x) = \frac{G(x)}{x} = \int_0^\infty (1+u)^{a-1}e^{-xu}du, \quad (60)$$

which gives

$$-R'(x) = \frac{1}{x^2}\int_0^\infty (u+1)^{a-1}\left[x^2ue^{-xu}\right]du < 0. \quad (61)$$

The expression in brackets of (61) is a probability density function. For $a \geq 2$ or $a \leq 1$, $(uh)^{a-1}$ is convex function of $u \geq 0$. So Jensen's inequality gives,

$$-R'(x) \geq \frac{1}{x^2}\left(1+\frac{2}{x}\right)^{a-1}. \quad (62)$$

Since

$$R'(x) = \left(\frac{x-a}{x}\right)R(x) - \frac{1}{x}, \quad (63)$$

we get

$$\left(\frac{x-a}{x}\right)R(x) < \frac{1}{x} - \frac{\left(1+\frac{2}{x}\right)^{a-1}}{x^2}, \quad (64)$$

which gives for $x > a$,

$$R(x) < \frac{1}{x-a}\left(1-\frac{\left(1+\frac{2}{x}\right)^{a-1}}{x}\right). \quad (65)$$

By the definition of $R(x)$, this leads to (56) and (58). Simply reverse the inequality sign to prove (59) using concavity instead of convexity. This completes the proof. ∎

Remark 1. Note that when $a = 1$ or $a = 2$ in Theorem 2.6, equality holds. Thus, $\Gamma(a,x) = M_1(a,x)$ in these cases.

Next, let us construct rational function approximations to $\Gamma(a,x)$ and then investigate conditions under which these approximations are upper or lower bounds for $\Gamma(a,x)$. Since

$$G(x) = H(a,x) = \frac{\int_x^\infty t^{a-1}e^{-t}dt}{x^{a-1}e^{-x}}, \tag{66}$$

from (44) differentiation of $G(x)$ with respect to x gives

$$G'(x) = \left(1 - \left(\frac{a-1}{x}\right)\right)G(x) - 1. \tag{67}$$

Let $R(x) = \frac{G(x)}{x}, x > 0$. Then using (67) and $G(x) = \frac{R(x)}{x}$, we get

$$R'(x) = \frac{xG'(x) - G(x)}{x^2} = \left(\frac{x-a}{x}\right)R(x) - \frac{1}{x}. \tag{68}$$

Also, $G(x) = x\int_0^\infty (u+1)^{a-1}e^{-xu}du$. Thus,

$$R(x) = \int_0^\infty (u+1)^{a-1}e^{-xu}du. \tag{69}$$

The derivatives of $R(x)$ are:

$$R^{(k)}(x) = (-1)^k \int_0^\infty (u+1)^{a-1}u^k e^{-xu}du, \quad k = 1,2,3,\dots\ . \tag{70}$$

Thus, $R^{(k)}(x) > 0$, if k is even and $R^{(k)}(x) < 0$, if k is odd. From (70)

$$R'(x) = p_1(x)R(x) + q_1(x) < 0, \tag{71}$$

where

$$p_1(x) = \frac{x-a}{x} \quad \text{and} \quad q_1(x) = \frac{-1}{x}. \tag{72}$$

Similarly,

$$R''(x) = p_2(x)R(x) + q_2(x) > 0 \tag{73}$$

where

$$p_2(x) = (x-a)^2 + a \quad \text{and} \quad q_2(x) = x(x-a-1). \tag{74}$$

A simple inductive argument shows that

$$R^{(k)}(x) = p_k(x)R(x) + q_k(x), \quad k = 2,3,4,\dots \tag{75}$$

where the $\{p_k(x)\}$ and $\{q_k(x)\}$ rational functions are generated recursively according to:

$$p_{k+1}(x) = p_k(x)p_1(x) + p_k'(x), \quad k = 1,2,3,\dots \tag{76}$$

and

$$q_{k+1}(x) = p_k(x)q_1(x) + q_k'(x), \quad k = 1,2,3,\dots. \tag{77}$$

Thus,

$$R^{(k)}(x) = p_k(x)R(x) + q_k(x) < 0 \quad \text{if } k \text{ is odd}, \tag{78}$$

and

$$R^{(k)}(x) = p_k(x)R(x) + q_k(x) > 0 \quad \text{if } k \text{ is even}. \tag{79}$$

Inequalities (78) and (79) can be solved to get bounds on $R(x)$, hence on $G(x)$ and $\Gamma(a,x)$, provided we can determine, for each value of k, when:

$$p_k(x) > 0 \quad \text{and} \quad q_k(x) < 0. \tag{80}$$

These conditions satisfying (80) will determine the inequalities that involve x and a which are sufficient to give either an upper bound or a lower bound for $\Gamma(a,x)$. Let us list some of the $p_k(x)$ and $q_k(x)$ rational functions for $k \le 4$:

$$k = 1: \ p_1(x) = \frac{x-a}{a}, \quad q_1(x) = \frac{-1}{x} \tag{81}$$

$$k = 2: \ p_2(x) = \frac{(x-a)^2 + a}{x^2}, \quad q_2(x) = \frac{-(x-a-1)}{x^2} \tag{82}$$

$$k = 3: \ p_3(x) = \frac{-a^3 + (3x-3)a^2 + (-3x^2+3x-2)a + x^3}{x^3}, \tag{83a}$$

$$q_3(x) = \frac{-a^2 + 2ax - x^2 - 3a + x - 2}{x^3}. \tag{83b}$$

$$k = 4: \; p_4(x) = \frac{a^4 + (6-4x)a^3 + (6x^2 - 12x + 11)a^2 + (-4x^3 + 6x^2 - 8x + 6)a + x^4}{x^4}, \tag{84a}$$

$$q_4(x) = \frac{(-x^3 + (3a+1)x^2 + (-3a^2 - 7a - 2)x + (a^3 + 6a^2 + 11a + 6))}{x^4}. \tag{84b}$$

For each value of k, we must find values of x and a satisfying:

$$R(x) < \frac{-q_k(x)}{p_k(x)} \quad \text{if } k \text{ is odd,} \tag{85}$$

and

$$R(x) > \frac{-q_k(x)}{p_k(x)} \quad \text{if } k \text{ is even.} \tag{86}$$

It was attempted to find an algorithm to do this which would work for all k, but this was unsuccessful. What is true is that as $x \longrightarrow \infty$, both (85) and (86) hold. Thus, we shall examine the first several values of k on an individual basis to obtain our next couple of bounds for $\Gamma(a, x)$. From the relation between $\Gamma(a, x)$ and $R(x)$, we would then have:

$$\Gamma(a, x) < x^a e^{-x} \left(\frac{-q_k(x)}{p_k(x)} \right) \quad \text{as } x \longrightarrow \infty \quad \text{if} \quad k \text{ is odd,} \tag{87}$$

and

$$\Gamma(a, x) > x^a e^{-x} \left(\frac{-q_k(x)}{p_k(x)} \right) \quad \text{as } x \longrightarrow \infty \quad \text{if} \quad k \text{ is even.} \tag{88}$$

For $k = 1$, the analysis is simple. We merely state the following theorem.

Theorem 2.7. *For a real and* $x > \max(a, 0)$*,*

$$\begin{aligned} \Gamma(a, x) &\le x^a e^{-x} \left(\frac{-q_1(x)}{p_1(x)} \right) \\ &= x^a e^{-x} \cdot \left(\frac{1}{x - a} \right). \end{aligned} \tag{89}$$

The theorem below gives a nontrivial (positive) lower bound on $\Gamma(a, x)$ when k = 2.

Theorem 2.8.

$$\begin{aligned} \Gamma(a, x) &> x^a e^{-x} \left(\frac{-q_2(x)}{p_2(x)} \right) \\ &= x^a e^{-x} \left(\frac{x - (a+1)}{(x-a)^2 + a} \right), \end{aligned} \tag{90}$$

holds if either (a) or (b) holds.

(a) $a \ge 0$ *and* $x > a + 1$
(b) $a < 0$ *and* $x > \max\{a + \sqrt{-a}, a + 1, 0\}$.

Proof. From (75), $R''(x) = p_2(x)R(x) + q_2(x) > 0$ which gives

$$\Gamma(a, x) > x^a e^{-x} \left(\frac{-q_2(x)}{p_2(x)} \right) = x^a e^{-x} \left(\frac{x - (a+1)}{(x-a)^2 + a} \right), \tag{91}$$

provided $q_2(x) < 0$ and $p_2(x) > 0$. Clearly $p_2(x) > 0$ for $a \ge 0$. If $a \ge 0$, $q_2(x) < 0$ if $x > a + 1$ and $x > 0$ which gives $x > a + 1$. If $a < 0$, we require $p_2(x) < 0$ which requires either $x > a = \sqrt{-a}$ or $x < a - \sqrt{-a} < 0$. The latter is impossible as $x > 0$. So $x > a + \sqrt{-a}$ must hold as well as $x > a + 1$ to ensure $q_2(x) > 0$ and a nontrivial lower bound for $R(x)$. Thus, if $x > \max\{a + \sqrt{-a}, a + 1, 0\}$, then $R(x) > \frac{-q_2(x)}{p_2(x)}$, which gives

$$\Gamma(a, x) > x^a e^{-x} \left(\frac{-q_2(x)}{p_2(x)} \right) = x^a e^{-x} \left(\frac{x - (a+1)}{(x-a)^2 + a} \right). \tag{92}$$

This completes the proof. ■

From (81)–(84b), we see that for $k = 1, 2, 3, 4$:

$$\begin{aligned} x^k p_k(x) &= x^k + q_{k-1}(a, x) \quad \text{and} \\ -x^k q_k(x) &= x^{k-1} + h_{k-1}(a, x), \end{aligned} \tag{93}$$

where $q_{k-1}(a,x)$ is polynomial in x of degree $k-1$ and $h_{k-1}(a,x)$ is a polynomial in x of degree $k-2$. This is easily proven by a simple induction argument. Thus,

$$\frac{-q_k(x)}{p_k(x)} \sim \frac{1}{x} \quad \text{as} \quad x \longrightarrow \infty. \tag{94}$$

Also, as $x \longrightarrow \infty$,

$$\Gamma(a,x) < x^a e^{-x}\left(\frac{-q_k(x)}{p_k(x)}\right) \quad \text{as } x \longrightarrow \infty \quad \text{if } k \quad \text{is odd} \tag{95}$$

and

$$\Gamma(a,x) > x^a e^{-x}\left(\frac{-q_k(x)}{p_k(x)}\right) \quad \text{as } x \longrightarrow \infty \quad \text{if } k \quad \text{is even.} \tag{96}$$

Thus. for each odd k we must solve for a function $x_U(a)$ such that

$$\Gamma(a,x) < x^a e^{-x}\left(\frac{-q_k(x)}{p_k(x)}\right) \qquad \text{for } x > x_U(a). \tag{97}$$

and for each even k we must solve for a function $x_L(a)$ such that

$$\Gamma(a,x) > x^a e^{-x}\left(\frac{-q_k(x)}{p_k(x)}\right) \qquad \text{for } x > x_L(a). \tag{98}$$

Often, $x_U(a)$ and $x_L(a)$ will be the only real positive root of $p_k(x)=0$ or perhaps the largest real positive root if $a>0$. For $k=1$, Theorem 2.7 found $x_U(a)=\max(a,0)$. For k For $k=2$, $x_U(a)=a+1$ for real a, Theorem 2.8 found

$$x_L(a) = \begin{cases} a+1 & if \quad a \geq 0 \\ \max\{a+\sqrt{a}, a+1, 0\} & if \quad a < 0. \end{cases}$$

For arbitrary values of $k \geq 3$, to determine $x_U(a)$ and $x_L(a)$ would be difficult. We would need to solve the inequalities $p_k(x)>0$ and $q_k(x)<0$ for x to get $x_U(a)$ and $x_L(a)$. This would involve finding formulas for roots (largest root) of the polynomials of arbitrary degree k, a very challenging task. Next, we present the upper bound on $\Gamma(a,x)$ for $k=3$ when $a>0$.

Theorem 2.9. *Suppose* $a>0$. *Let*

$$x_U(a) = -2\sqrt{a}\sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{-1}{\sqrt{a}}\right)\right] \tag{99a}$$

$$= \sqrt[3]{a+\sqrt{a^2+a^3}} - \sqrt[3]{\sqrt{a^2+a^3}-a}. \tag{99b}$$

Then for $x > x_U(a)$,

$$\Gamma(a,x) < x^a e^{-x}\left(\frac{-q_3(x)}{p_3(x)}\right), \tag{100}$$

where $p_3(x)$ and $q_3(x)$ are given by (83a) and (83b), respectively. In particular, (100) holds if

$$x > a + \frac{2}{3}. \tag{101}$$

Proof. We apply classical methods for solving cubic polynomials. A cubic polynomial with real coefficients of the form $a_1x^3+a_2x^2+a_3x+a_4$, $a_1 \neq 0$, will have a unique real root if $4p^3+27q^2>0$ and $p>0$, where

$$p = \frac{3a_1a_3 - a_2^2}{3a_1^2} \quad \text{and} \quad q = \frac{2a_2^3 - 9a_1a_2a_3 + 27a_1^2a_4}{27a_1^3}. \tag{102}$$

Applying this to the numerator of $p_3(x)$ with $a_1=1, a_2=-3a, a_3=3a^2+3a$ and $a_4=-a^3-3a^2-2a$. We get, after simplification, $p=3a$ and $q=-2a$. Clearly, $p>0$. Also, the discriminant of the cubic polynomial $=4p^3+27q^2=a^3+a^2>0$ since $a>0$. Thus, $p_3(x)$ has one real root and two complex roots. In the case where $4p^3+27q^2>0$ and $p>0$, this single real root is given by

$$\begin{aligned} x &= \frac{-a_2}{3a_1} + \left(\frac{-q}{2}+\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}\right)^{1/3} + \left(\frac{-q}{2}-\sqrt{\frac{p^3}{27}+\frac{q^2}{4}}\right)^{1/3} \\ &= a + \left(a+\sqrt{a^2+a^3}\right)^{1/3} + \left(a-\sqrt{a^2+a^3}\right)^{1/3} \\ &= a + \left(a+\sqrt{a^2+a^3}\right)^{1/3} - \left(\sqrt{a^2+a^3}-a\right)^{1/3} \\ &= -2\sqrt{a}\sinh\left[\frac{1}{3}\operatorname{arcsinh}\left(\frac{-1}{\sqrt{a}}\right)\right] \\ &= x_U(a). \end{aligned} \tag{103}$$

Also, it is easily verified that $q_3(x)$ has no real roots and is strictly negative for $a > 0$. Thus, from (103), (99a)–(99b) hold. To prove that (100) holds, it suffices to show that $a + \frac{2}{3} > x_U(a)$. Let $c = \sqrt{a^2 + a^3} + a$ and $d = \sqrt{a^2 + a^3} - a$. Then

$$x_U(a) - a = c^{1/3} - d^{1/3} = \frac{c - d}{c^{2/3}c^{1/3} + d^{1/3} + d^{2/3}} = \frac{\frac{1}{3}(c - d)}{\frac{1}{3}\left(c^{2/3} + c^{1/3}d^{1/3} + d^{2/3}\right)}. \tag{104}$$

Applying the arithmetic mean/geometric mean inequality to the denominator of (104), we get, since $c > d > 0$, that

$$\frac{1}{3}\left(c^{2/3} + c^{1/3}d^{1/3} + d^{2/3}\right) > (cd)^{1/3} = a. \tag{105}$$

Thus, from (104), we get

$$x_U(a) - a < \frac{2a/3}{a} = \frac{2}{3}. \tag{106}$$

Thus, $x_U(a) < a + \frac{2}{3}$ holds. Note that $\frac{2}{3}$ cannot be replaced by a smaller constant, since $\lim_{a\longrightarrow\infty}\left(x_U(a) - a\right) = \frac{2}{3}$, by a simple computation. This proves (100) and completes the proof of the Theorem. ∎

Next, we investigate the lower bound on $\Gamma(a, x)$ corresponding to $k = 4$. This will be the last case we investigate individually.

Theorem 2.10. *Suppose $a > 0$. Let $b_1 = \frac{5(a+1)}{3}$ and $b_2 = -\frac{22a}{3} - \frac{146}{27}$. Let*

$$x_L(a) = \left(a + \frac{1}{3}\right) + \left(\frac{-b_2}{2} + \sqrt{\frac{b_1^3}{27} + \frac{b_2^2}{4}}\right)^{1/3} - \left(\frac{b_2}{2} + \sqrt{\frac{b_1^3}{27} + \frac{b_2^2}{4}}\right)^{1/3}. \tag{107}$$

Then for $x > x_L(a)$, we have

$$\Gamma(a, x) > x^a e^{-x}\left(\frac{-q_4(x)}{p_4(x)}\right), \tag{108}$$

where $p_4(x)$ and $q_4(x)$ are given by (84a) and (84b), respectively. Also

$$x_L(a) < \left(a + \frac{1}{3}\right) + \frac{198a + 146}{135a + 145} < a + \frac{161}{45}. \tag{109}$$

Thus, (107) holds, in particular, if $x > a + \frac{161}{45}$

Proof. We shall apply standard methods for determining the nature of solutions to the quartic equation:

$$a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5 = 0 \tag{110}$$

where the a_i's are real with $a_1 \neq 0$. Let $p = 8a_1a_3 - 3a_2^2, D = 64a_1^3a_5 - 16a_1^2a_3^2 + 16a_1a_2^2a_3 - 16a_1^2a_2a_4 - 3a_2^4$. Then for quartic (110), there will be no real roots if either $P > 0$ or $D > 0$. Here, the quartic polynomial is the numerator of $p_4(x)$ in (84a) with: $a_1 = 1, a_2 = -4a, a_3 = 6a^2 + a, a_4 = -4a^3 + 2a^2 - 8a$ and $a_5 = a^4 + 6a^3 + 11a^2 + 6a$. We obtain $P = 8a > 0$ and $p_4(x)$ has no real roots. Also, $p_4(0) = a^4 + 6a^3 + 11a^2 + 6a > 0$, which gives $p_4(x) > 0$ for all $x > 0$ and $a > 0$, since $p_4(x) = 0$ has no real roots. Now consider the roots of $q_4(x)$. Then the numerator of $q_4(x)$ in (84b) is

$$-x^3 + (3a + 1)x^2 - (3a^2 + 7a + 2)x + (a^3 + 6a^2 + 11a + 6).$$

Applying the same method for solving cubics used earlier in the proof of Theorem 2.9 and proceeding as done there, we obtain the following, single simple real root of $q_4(x)$ as

$$x = x_L(a) = \left(a + \frac{1}{3}\right) + \left(\frac{-b_2}{2} + \sqrt{\frac{b_1^3}{27} + \frac{b_2^2}{4}}\right)^{1/3} - \left(\frac{b_2}{2} + \sqrt{\frac{b_1^3}{27} + \frac{b_2^2}{4}}\right)^{1/3}. \tag{111}$$

Thus, for $x > x_L(a)$, we have

$$\Gamma(a, x) > x^a e^{-x}\left(\frac{-q_4(x)}{p_4(x)}\right). \tag{112}$$

Letting

$$c = \left(\frac{-b_2}{2} + \sqrt{\frac{b_1^3}{27} + \frac{b_2^2}{4}}\right)^{1/3} \quad \text{and} \quad d = \left(\frac{b_2}{2} + \sqrt{\frac{b_1^3}{27} + \frac{b_2^2}{4}}\right)^{1/3} \tag{113}$$

and using the arithmetic mean/geometric mean inequality again as done in Theorem 2.9, we get

$$\begin{aligned} x_L(a) &< \left(a+\frac{1}{3}\right) + \frac{\frac{1}{3}(c-d)}{(cd)^{1/3}} = \left(a+\frac{1}{3}\right) - \frac{b_2}{b_1} \\ &= \left(a+\frac{1}{3}\right) + \left(\frac{198a+146}{135a+145}\right) \\ &< \left(a+\frac{1}{3}\right) + \frac{146}{145} = a + \frac{161}{45}. \end{aligned} \tag{114}$$

This completes the proof. ■

Next, let us consider some perturbations of some of the rational function bounds given earlier which will be improvements of these bounds and also be valid bounds for the same set of positive $x-$ values.

Theorem 2.11. *Suppose* $0 < a \le 1$ *or* $a \ge 2$. *Let*

$$M_k(a,x) = (-1)^k (k!) \left(\frac{1}{x}\right)^{k+1} \left(1 + \frac{k+1}{x}\right)^{a-1}, \quad k = 1,2,3,\ldots \tag{115}$$

Then

$$\Gamma(a,x) \le x^a e^{-x} \left(\frac{M_k(a,x) - q_k(x)}{p_k(x)}\right) \quad \text{if } x > x_U(a), \quad k \text{ odd}, \tag{116}$$

and

$$\Gamma(a,x) \ge x^a e^{-x} \left(\frac{M_k(a,x) - q_k(x)}{p_k(x)}\right) \quad \text{if } x > x_L(a), \quad k \text{ even}, \tag{117}$$

where $x_U(a)$ *and* $x_L(a)$ *were given earlier in Theorems 2.8–2.10. If* $1 < a < 2$, *then (116) and (117) hold with the inequality signs reversed in the first inequality.*

Proof. From (75), we have, for $0 < a \le 1$ or $a \ge 2$, :

$$R^{(k)}(x) = p_k(x) R(x) + q_k(x), \tag{118}$$

where

$$R^{(k)}(x) = (-1)^k \int_0^\infty (1+u)^{a-1} u^k e^{-xu} du. \tag{119}$$

Then

$$R^{(k)}(x) = (-1)^k \left(\frac{1}{x}\right)^{k+1} k! \int_0^\infty (1+u)^{a-1} \left[\frac{x^{k+1}}{k!} u^k e^{-xu}\right] du. \tag{120}$$

The expression in brackets of (120) is a probability density function on $[0,\infty)$. Also, $(1+u)^{a-1}$ is a convex function of u. By Jensen's inequality, we get

$$\int_0^\infty (1+u)^{a-1} \left[\frac{x^{k+1}}{k!} u^k e^{-xu}\right] du \ge \left(1 + \frac{k+1}{x}\right)^{a-1}. \tag{121}$$

From (121), this gives

$$\begin{aligned} &R^{(k)}(x) \le M_k(a,x) \quad if \;\; k \;\; is\ odd\ and \\ &R^{(k)}(x) \ge M_k(a,x) \quad if \;\; k \;\; is\ even. \end{aligned} \tag{122}$$

Thus, (118) gives

$$p_k(x) R(x) + q_k(x) \le M_k(a,x) \quad if \;\; k \;\; is\ odd \tag{123}$$

and

$$p_k(x) R(x) + q_k(x) \ge M_k(a,x) \quad if \;\; k \;\; is\ even. \tag{124}$$

Solving for $R(x)$ in (123)–(124), we obtain:

$$R(x) \le \frac{M_k(a,x) - q_k(x)}{p_k(x)} \quad if \;\; k \;\; is\ odd\ and\ x > x_U(a), \tag{125}$$

and

$$R(x) \ge \frac{M_k(a,x) - q_k(x)}{p_k(x)} \quad if \;\; k \;\; is\ even\ and\ x > x_L(a). \tag{126}$$

Since $\Gamma(a,x) = x^a e^{-x} R(x)$, (116) and (117) follow form (125) and (126), respectively. This completes the proof. In the case $0 < a \le 1$ or $a \ge 2$. For $1 < a < 2$, the above proof applies with 'both' inequality signs reversed since $(1+u)^{a-1}$ is a concave function of u instead. ■

Theorem 2.12. *Suppose $a > 0$ and $x > 0$. Let*

$$N_k(a,x) = \frac{(-1)^{k+1}(k!)}{x^k}. \tag{127}$$

(a). Suppose that either ($a \geq 1$ and k is odd) or ($0 < a < 1$ and k is even). Then

$$\Gamma(a,x) < x^a e^{-x}\left(\frac{-q_k(x)}{p_k(x)+N_k(a,x)}\right) \quad if\ p_k(x)+N_k(a,x) > 0. \tag{128}$$

(b). If ($0 < a < 1$ and k is odd) or ($a \geq 1$ and k is even). Then

$$\Gamma(a,x) > x^a e^{-x}\left(\frac{-q_k(x)}{p_k(x)+N_k(a,x)}\right) \quad if\ p_k(x)+N_k(a,x) > 0. \tag{129}$$

Proof. We prove only the case $a \geq 1$ and k is odd. The other parts are proven in a very similar fashion and are omitted. Then

$$R^{(k)}(x) = \frac{(-1)^k}{x}\int_0^\infty (1+u)^{a-1}u^k\left[xe^{-xu}\right]du. \tag{130}$$

The expression in brackets of (130) is a probability density function. Since $f_1(u) = (u+1)^{a-1}$ and $f_2(u) = u^k$ are non-decreasing functions of u, standard covariance inequalities from probability theory give

$$\int_0^\infty (1+u)^{a-1}u^k\left[xe^{xu}\right]du \geq \left(\int_0^\infty (u+1)^{a-1}xe^{-xu}du\right)\cdot\left(\int_0^\infty u^k xe^{-xu}du\right), \tag{131}$$

which gives, from (130), that

$$R^{(k)}(x) \leq \frac{(-1)^k k!}{x^k}\cdot R(x). \tag{132}$$

So,

$$p_k(x)R(x) + q_k(x) < N_k(a,x)R(x). \tag{133}$$

Solving for $R(x)$ in (133), we get

$$\Gamma(a,x) < x^a e^{-x}\left(\frac{-q_k(x)}{p_k(x)+N_k(a,x)}\right) \quad if\ p_k(x)+N_k(a,x) > 0. \tag{134}$$

Thus,

$$\begin{aligned}\Gamma(a,x) &= x^a e^{-x}R(x)\\ &< x^a e^{-x}\left(\frac{-q_k(x)}{p_k(x)+N_k(a,x)}\right) \quad if\ p_k(x)+N_k(a,x) > 0.\end{aligned} \tag{135}$$

This completes the proof. ■

Remark 2. If k is odd, then bounds (128) and (129) hold on a smaller subinterval of $(0,\infty)$ than the bounds given earlier in Theorems. Also, when k is even, a bound of the 'opposite' type is provided (an upper bound for $0 < a < 1$) than the bound type (lower) given in the Theorem earlier. A similar remark holds for k even. Note that $\lim_{x\longrightarrow\infty}\left[p_k(x)+N_k(a,x)\right] = \infty$, so that there will exist a positive real number $x_k(a)$ such that $x > x_k(a)$ implies $p_k(x)+N_k(a,x) > 0$, so that Theorem 2.12 is applicable.

3. Numerical results

In this section, we conduct an empirical study to evaluate the performance of the proposed bounds. We calculate the exact values of $\Gamma(a,x)$, the bounds for the proposed theorems, and their respective relative errors.

Table 1 compares the proposed upper bound from Theorem 2.6 with the existing upper bound from Theorem 1.1 [1] for the incomplete gamma function $\Gamma(a,x)$ across various values of x and a. From the result, we observe that Theorem 2.6 consistently provides a tighter upper bound than Theorem 1.1. The relative errors associated with Theorem 2.6 are significantly smaller in some cases by several orders of magnitude than those from Theorem 1.1. For example, when $x = 5$ and $a = 2$, the relative error for Theorem 1.1 is 4.17×10^{-2}, whereas for Theorem 2.6 it is only 2.15×10^{-7}. This indicates that Theorem 2.6 offers a much more accurate approximation of the incomplete gamma function's upper bound across all tested values of x and a. The results are graphed in Fig. 1.

Table 2 evaluates the proposed lower bound from Theorem 2.8 against the existing lower bound from Theorem 1.1 [1] for various values of x and a. The results show that Theorem 2.8 provides a significantly better lower bound than Theorem 1.1. For instance, at $x = 15$ and $a = 2$, the relative error for Theorem 1.1 is 6.25×10^{-2}, while for Theorem 2.8, it is only 1.31×10^{-2}, reflecting a substantial improvement. This pattern holds across all presented values, demonstrating that Theorem 2.8 yields more precise lower bounds for the incomplete gamma function. The results are graphed in Fig. 2.

Table 3 compares the upper bounds from Theorems 2.4 and 2.6 with the existing upper bound from Theorem 1.3 [1] for negative values of a (specifically $a = -4, -3, -2$) across various x values. The results indicate that Theorems 2.4 and 2.6 offer improved upper

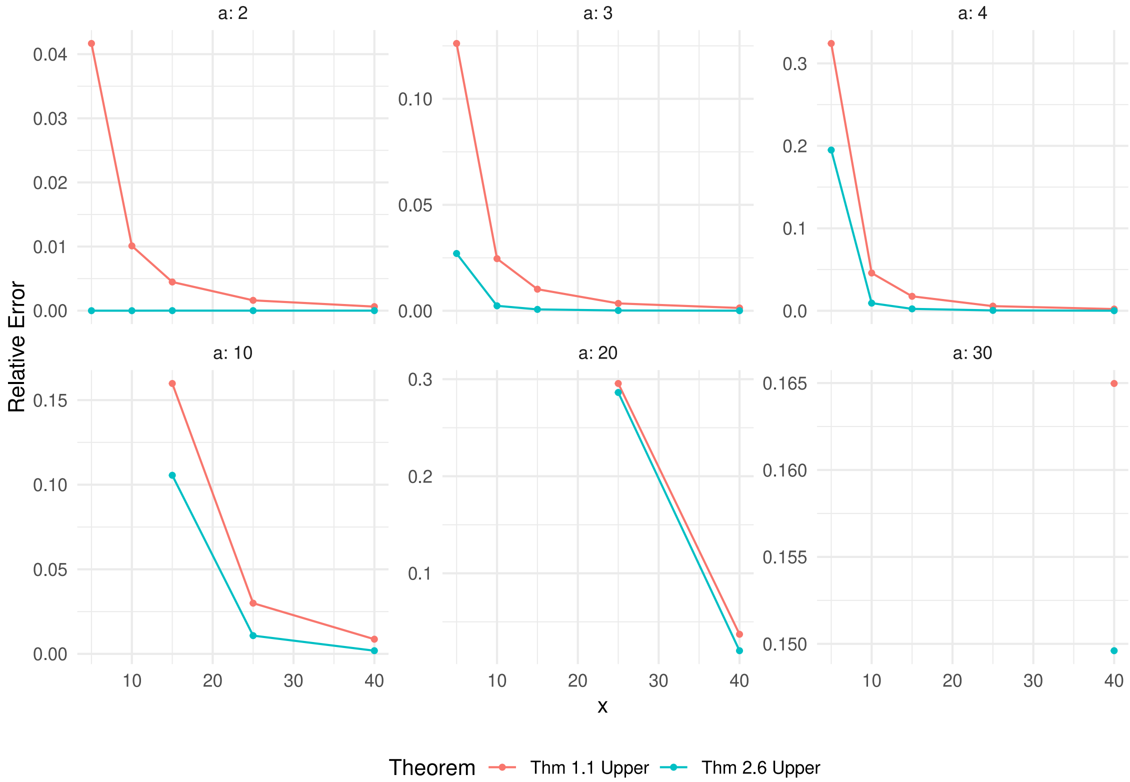

Fig. 1. Comparisons for Lower Bounds 1.1 [1] and 2.8.

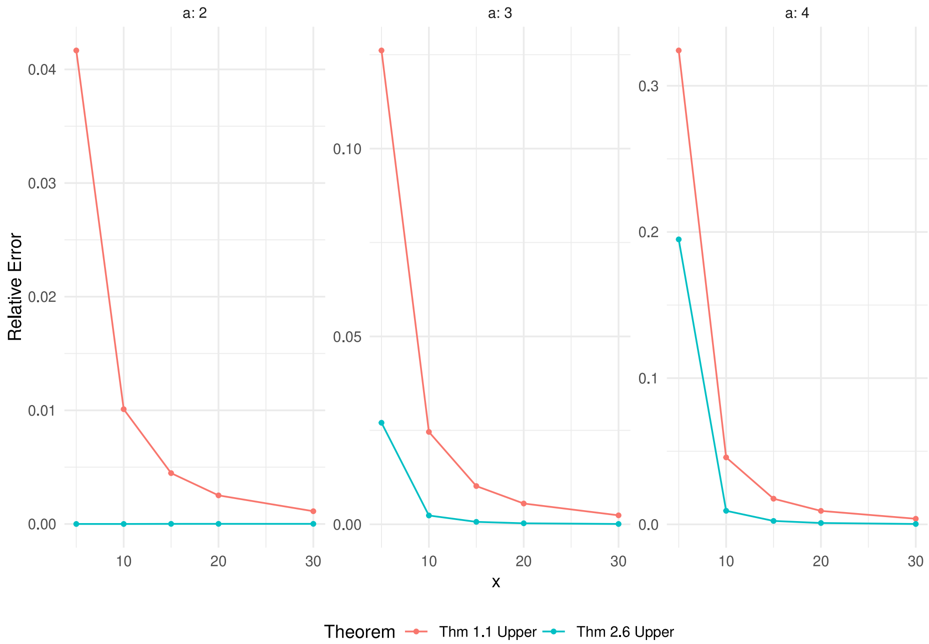

Fig. 2. Comparisons for Upper Bounds 1.1 [1] and 2.6.

Table 1
Comparisons for upper bounds 1.1 [1] and 2.6.

x	a	$\Gamma(a,x)$	Theorem 1.1	Theorem 2.6	RE(Theorem 1.1)	RE(Theorem 2.6)
5	2	4.042767×10^{-2}	4.211217×10^{-2}	4.042768×10^{-2}	4.166689×10^{-2}	2.15347×10^{-7}
	3	2.49304×10^{-1}	2.807478×10^{-1}	2.56042×10^{-1}	1.261261×10^{-1}	2.702703×10^{-2}
	4	1.590155×10^{0}	2.105608×10^{0}	1.900101×10^{0}	3.241525×10^{-1}	1.949153×10^{-1}
	10	3.513302×10^{5}	NA	NA	NA	NA
	20	1.216451×10^{17}	NA	NA	NA	NA
	30	8.841762×10^{30}	NA	NA	NA	NA
10	2	4.993991×10^{-4}	5.044437×10^{-4}	4.993992×10^{-4}	1.010123×10^{-2}	2.173759×10^{-7}
	3	5.538792×10^{-3}	5.674991×10^{-3}	5.551763×10^{-3}	2.459013×10^{-2}	2.341883×10^{-3}
	4	6.20163×10^{-2}	6.485704×10^{-2}	6.259137×10^{-2}	4.580633×10^{-2}	9.272825×10^{-3}
	10	1.661735×10^{5}	NA	NA	NA	NA
	20	1.212249×10^{17}	NA	NA	NA	NA
	30	8.84176×10^{30}	NA	NA	NA	NA
15	2	4.894388×10^{-6}	4.916287×10^{-6}	4.894437×10^{-6}	4.474324×10^{-3}	9.99324×10^{-6}
	3	7.861689×10^{-5}	7.941695×10^{-5}	7.866788×10^{-5}	1.017669×10^{-2}	6.48605×10^{-4}
	4	1.268271×10^{-3}	1.290525×10^{-3}	1.271219×10^{-3}	1.754679×10^{-2}	2.324014×10^{-3}
	10	2.53485×10^{4}	2.939978×10^{4}	2.802448×10^{4}	1.598235×10^{-1}	1.055676×10^{-1}
	20	1.064661×10^{17}	NA	NA	NA	NA
	30	8.838062×10^{30}	NA	NA	NA	NA
25	2	3.610828×10^{-10}	3.616652×10^{-10}	3.610865×10^{-10}	1.613069×10^{-3}	1.048812×10^{-5}
	3	9.402062×10^{-9}	9.434744×10^{-9}	9.403401×10^{-9}	3.476074×10^{-3}	1.423438×10^{-4}
	4	2.452055×10^{-7}	2.465899×10^{-7}	2.453153×10^{-7}	5.645833×10^{-3}	4.47767×10^{-4}
	10	8.036944×10^{1}	8.27786×10^{1}	8.123691×10^{1}	2.997598×10^{-2}	1.07935×10^{-2}
	20	1.624872×10^{16}	2.105169×10^{16}	2.090109×10^{16}	2.955901×10^{-1}	2.863219×10^{-1}
	30	7.231643×10^{30}	NA	NA	NA	NA
40	2	1.741806×10^{-16}	1.742915×10^{-16}	1.741825×10^{-16}	6.361754×10^{-4}	1.077781×10^{-5}
	3	7.145663×10^{-15}	7.155123×10^{-15}	7.145961×10^{-15}	1.323854×10^{-3}	4.175398×10^{-5}
	4	2.933296×10^{-13}	2.939402×10^{-13}	2.933621×10^{-13}	2.081506×10^{-3}	1.106584×10^{-4}
	10	1.424644×10^{-3}	1.437007×10^{-3}	1.427318×10^{-3}	8.678385×10^{-3}	1.877224×10^{-3}
	20	2.144638×10^{13}	2.22434×10^{13}	2.188012×10^{13}	3.71634×10^{-2}	2.022399×10^{-2}
	30	3.822177×10^{29}	4.452745×10^{29}	4.393996×10^{29}	1.64976×10^{-1}	1.496056×10^{-1}

Table 2
Comparisons for lower bounds 1.1 [1] and 2.8.

x	a	$\Gamma(a,x)$	Theorem 1.1	Theorem 2.8	RE (Theorem 1.1)	RE (Theorem 2.8)
15	2	4.894388×10^{-6}	4.588535×10^{-6}	4.830037×10^{-6}	6.249063×10^{-2}	1.314803×10^{-2}
	3	7.861689×10^{-5}	6.882802×10^{-5}	7.725594×10^{-5}	1.245135×10^{-1}	1.731111×10^{-2}
	5	2.055939×10^{-2}	1.54863×10^{-2}	1.991096×10^{-2}	2.467527×10^{-1}	3.153914×10^{-2}
	8	$9.073105\times10^{+1}$	$5.226628\times10^{+1}$	$8.25257\times10^{+1}$	4.239428×10^{-1}	9.043595×10^{-2}
20	2	4.328378×10^{-8}	4.122307×10^{-8}	4.299339×10^{-8}	4.760924×10^{-2}	6.709022×10^{-3}
	3	9.110298×10^{-7}	8.244614×10^{-7}	9.035194×10^{-7}	9.502249×10^{-2}	8.243826×10^{-3}
	5	4.06674×10^{-4}	3.297846×10^{-4}	4.014769×10^{-4}	1.89069×10^{-1}	1.27796×10^{-2}
	8	$3.924094\times10^{+0}$	$2.638277\times10^{+0}$	$3.818558\times10^{+0}$	3.276724×10^{-1}	2.689429×10^{-2}
30	2	2.900832×10^{-12}	2.807287×10^{-12}	2.893006×10^{-12}	3.224779×10^{-2}	2.698105×10^{-3}
	3	9.001954×10^{-11}	8.421861×10^{-11}	8.974114×10^{-11}	6.444083×10^{-2}	3.092689×10^{-3}
	5	8.698323×10^{-8}	7.579675×10^{-8}	8.662485×10^{-8}	1.28605×10^{-1}	4.120023×10^{-3}
	8	2.637804×10^{-3}	2.046512×10^{-3}	2.620534×10^{-3}	2.241608×10^{-1}	6.547349×10^{-3}
40	2	1.741806×10^{-16}	1.699342×10^{-16}	1.739299×10^{-16}	2.437973×10^{-2}	1.439833×10^{-3}
	3	7.145663×10^{-15}	6.797367×10^{-15}	7.134263×10^{-15}	4.874234×10^{-2}	1.595457×10^{-3}
	5	1.204911×10^{-11}	1.087579×10^{-11}	1.202526×10^{-11}	9.737866×10^{-2}	1.979652×10^{-3}
	8	8.386611×10^{-7}	6.960504×10^{-7}	8.363396×10^{-7}	1.700458×10^{-1}	2.768158×10^{-3}

bounds over Theorem 1.3. The relative errors for Theorems 2.4 and 2.6 are consistently lower than those for Theorem 1.3, signifying greater accuracy. For example, when $a = -4$ and $x = 5$, the relative error for Theorem 1.3 is 6.08×10^{-2}, whereas for Theorem 2.6, it is 2.13×10^{-2}, and for Theorem 2.4, it is 4.15×10^{-2}. This improvement is evident across all the combinations of a and x, indicating the effectiveness of Theorems 2.4 and 2.6 in providing tighter upper bounds for negative a. The results are graphed in Fig. 3.

Table 4 presents a comparison between the proposed lower bound from Theorem 2.10 and the existing lower bound from Theorem 1.7 [1] for large x values ($x = 40$ to 90) and small integer values of a ($a = 1$ to 4). The results suggest that Theorem 2.10 generally provides lower bounds with accuracy comparable to or better than that of Theorem 1.7. For instance, at $x = 60$ and $a = 1$, the relative error for Theorem 1.7 is 6.56×10^{-4}, whereas for Theorem 2.10, it is 2.66×10^{-4}, indicating a more precise approximation by Theorem 2.10. While in some cases the relative errors are close, Theorem 2.10 tends to have smaller relative errors, especially for larger x values. This demonstrates that Theorem 2.10 effectively provides accurate lower bounds for the incomplete gamma function when x is large. The results are graphed in Fig. 4.

Table 3
Comparisons for upper bounds 1.3 [1], 2.4 and 2.6.

a	x	$\Gamma(a,x)$	Theorem 1.3	Theorem 2.6	Theorem 2.4	RE (Theorem 1.3)	RE (Theorem 2.6)	RE (Theorem 2.4)
−4	5	1.129218×10^{-6}	1.197857×10^{-6}	1.153313×10^{-6}	1.176078×10^{-6}	6.078516×10^{-2}	2.133786×10^{-2}	4.149816×10^{-2}
	10	3.089731×10^{-10}	3.242852×10^{-10}	3.112529×10^{-10}	3.121245×10^{-10}	4.955816×10^{-2}	7.378774×10^{-3}	1.019973×10^{-2}
	15	3.056829×10^{-13}	3.180271×10^{-13}	3.066878×10^{-13}	3.069214×10^{-13}	4.038229×10^{-2}	3.287464×10^{-3}	4.051483×10^{-3}
	20	5.191941×10^{-16}	5.367588×10^{-16}	5.200945×10^{-16}	5.202431×10^{-16}	3.383059×10^{-2}	1.734217×10^{-3}	2.020418×10^{-3}
	25	1.191374×10^{-18}	1.225970×10^{-18}	1.192595×10^{-18}	1.192750×10^{-18}	2.903851×10^{-2}	1.024653×10^{-3}	1.154881×10^{-3}
	30	3.313631×10^{-21}	3.397830×10^{-21}	3.315806×10^{-21}	3.316030×10^{-21}	2.540966×10^{-2}	6.563950×10^{-4}	7.238742×10^{-4}
	40	3.696563×10^{-26}	3.771621×10^{-26}	3.697742×10^{-26}	3.697829×10^{-26}	2.030484×10^{-2}	3.189497×10^{-4}	3.423522×10^{-4}
	50	5.619853×10^{-31}	5.714814×10^{-31}	5.620871×10^{-31}	5.620928×10^{-31}	1.689749×10^{-2}	1.811818×10^{-4}	1.913210×10^{-4}
−3	5	6.263850×10^{-6}	6.737947×10^{-6}	6.387158×10^{-6}	6.468429×10^{-6}	7.568780×10^{-2}	1.968573×10^{-2}	3.266029×10^{-2}
	10	3.304100×10^{-9}	3.492302×10^{-9}	3.323885×10^{-9}	3.329328×10^{-9}	5.696032×10^{-2}	5.988079×10^{-3}	7.635502×10^{-3}
	15	4.819769×10^{-12}	5.035429×10^{-12}	4.831952×10^{-12}	4.834012×10^{-12}	4.474502×10^{-2}	2.527919×10^{-3}	2.955218×10^{-3}
	20	1.080537×10^{-14}	1.120192×10^{-14}	1.081937×10^{-14}	1.082106×10^{-14}	3.669906×10^{-2}	1.295089×10^{-3}	1.451292×10^{-3}
	25	3.078742×10^{-17}	3.174387×10^{-17}	3.081056×10^{-17}	3.081272×10^{-17}	3.106641×10^{-2}	7.518249×10^{-4}	8.217939×10^{-4}
	30	1.022708×10^{-19}	1.050238×10^{-19}	1.023195×10^{-19}	1.023232×10^{-19}	2.691874×10^{-2}	4.763965×10^{-4}	5.122573×10^{-4}
	40	1.511636×10^{-24}	1.543733×10^{-24}	1.511983×10^{-24}	1.512001×10^{-24}	2.123357×10^{-2}	2.292842×10^{-4}	2.415455×10^{-4}
	50	2.861175×10^{-29}	2.911321×10^{-29}	2.861548×10^{-29}	2.861563×10^{-29}	1.752612×10^{-2}	1.304050×10^{-4}	1.356704×10^{-4}
−2	5	3.511203×10^{-5}	3.850255×10^{-5}	3.569625×10^{-5}	3.593572×10^{-5}	9.656291×10^{-2}	1.663850×10^{-2}	2.345872×10^{-2}
	10	3.548753×10^{-8}	3.783327×10^{-8}	3.564385×10^{-8}	3.567137×10^{-8}	6.610039×10^{-2}	4.404769×10^{-3}	5.180371×10^{-3}
	15	7.617799×10^{-11}	8.086735×10^{-11}	7.641467×10^{-11}	7.645137×10^{-11}	5.008496×10^{-2}	3.106394×10^{-3}	3.601323×10^{-3}
	20	2.280070×10^{-13}	2.385361×10^{-13}	2.283553×10^{-13}	2.283987×10^{-13}	4.606070×10^{-2}	1.526462×10^{-3}	1.718154×10^{-3}
	25	8.548143×10^{-16}	8.846705×10^{-16}	8.553774×10^{-16}	8.554424×10^{-16}	3.497303×10^{-2}	6.575344×10^{-4}	7.117656×10^{-4}
	30	3.672961×10^{-18}	3.780851×10^{-18}	3.674419×10^{-18}	3.674607×10^{-18}	2.938799×10^{-2}	3.964377×10^{-4}	4.220380×10^{-4}
	40	1.086350×10^{-22}	1.114507×10^{-22}	1.086717×10^{-22}	1.086735×10^{-22}	2.588874×10^{-2}	3.385672×10^{-4}	3.580977×10^{-4}
	50	3.323808×10^{-27}	3.402673×10^{-27}	3.323984×10^{-27}	3.323996×10^{-27}	2.330674×10^{-2}	2.026460×10^{-4}	2.079037×10^{-4}

Table 4
Comparisons for lower bounds 1.7 [1] and 2.10.

x	a	$\Gamma(a,x)$	Theorem 1.7	Theorem 2.10	RE (Theorem 1.7)	RE (Theorem 2.10)
40	1	4.248306×10^{-18}	4.246477×10^{-18}	4.245701×10^{-18}	4.306107×10^{-4}	6.133367×10^{-4}
	2	1.741806×10^{-16}	1.741482×10^{-16}	1.740626×10^{-16}	1.864861×10^{-4}	6.778576×10^{-4}
	3	7.145663×10^{-15}	7.144982×10^{-15}	7.140289×10^{-15}	9.528862×10^{-5}	7.521452×10^{-4}
	4	2.933296×10^{-13}	2.933316×10^{-13}	2.930838×10^{-13}	6.81055×10^{-6}	8.378869×10^{-4}
60	1	8.756412×10^{-27}	8.750666×10^{-27}	8.754079×10^{-27}	6.562095×10^{-4}	2.664238×10^{-4}
	2	5.341413×10^{-25}	5.339792×10^{-25}	5.339888×10^{-25}	3.035103×10^{-4}	2.85583×10^{-4}
	3	3.25914×10^{-23}	3.258554×10^{-23}	3.25814×10^{-23}	1.795907×10^{-4}	3.06805×10^{-4}
	4	1.989163×10^{-21}	1.988939×10^{-21}	1.988506×10^{-21}	1.125187×10^{-4}	3.303278×10^{-4}
70	1	3.975405×10^{-31}	3.972348×10^{-31}	3.974639×10^{-31}	7.690089×10^{-4}	1.927624×10^{-4}
	2	2.822538×10^{-29}	2.821519×10^{-29}	2.82196×10^{-29}	3.611299×10^{-4}	2.048782×10^{-4}
	3	2.004401×10^{-27}	2.00396×10^{-27}	2.003963×10^{-27}	2.197739×10^{-4}	2.181614×10^{-4}
	4	1.423698×10^{-25}	1.423492×10^{-25}	1.423367×10^{-25}	1.447875×10^{-4}	2.327281×10^{-4}
80	1	1.804831×10^{-35}	1.80324×10^{-35}	1.804569×10^{-35}	8.818083×10^{-4}	1.449474×10^{-4}
	2	1.461914×10^{-33}	1.461302×10^{-33}	1.46169×10^{-33}	4.184482×10^{-4}	1.531105×10^{-4}
	3	1.184331×10^{-31}	1.184024×10^{-31}	1.184139×10^{-31}	2.593059×10^{-4}	1.619956×10^{-4}
	4	9.596046×10^{-30}	9.594357×10^{-30}	9.594399×10^{-30}	1.760144×10^{-4}	1.71666×10^{-4}
90	1	8.19392×10^{-40}	8.18577×10^{-40}	8.193001×10^{-40}	9.946078×10^{-4}	1.12163×10^{-4}
	2	7.456469×10^{-38}	7.452923×10^{-38}	7.45559×10^{-38}	4.755647×10^{-4}	1.179365×10^{-4}
	3	6.786209×10^{-36}	6.784184×10^{-36}	6.785366×10^{-36}	2.984052×10^{-4}	1.24188×10^{-4}
	4	6.176961×10^{-34}	6.175685×10^{-34}	6.176152×10^{-34}	2.065537×10^{-4}	1.309551×10^{-4}

Table 5 compares the upper bounds obtained from Theorem 2.4 for various values of a and x where $a > x$. The relative errors between the upper bounds and the actual values are also presented. The relative errors associated with these bounds range from approximately 0.026 to 0.535. Notably, as a increases for a given x, the relative error generally decreases. This suggests improved accuracy of the upper bound with larger a. Further, it is important to note that the upper bounds for cases where $a > x$ is not addressed by Pinelis [1]. The results are graphed in Fig. 5.

Table 6 compares the actual values of the incomplete gamma function $\Gamma(a,x)$ with the bounds provided by Theorem 2.11 for different values of the parameters a and x. Specifically, it presents upper bounds when k is odd (here, $k = 1$) and lower bounds when k is even (here, $k = 2$), along with their corresponding relative errors. For $a = 0.5$ and increasing values of x, both the upper and lower bounds provided by Theorem 2.11 become increasingly accurate. The relative errors for $k = 1$ (upper bound) decrease from 2.739338×10^{-3} at $x = 5$ to 3.460803×10^{-5} at $x = 30$. Similarly, for $k = 2$ (lower bound), the relative errors decrease significantly, indicating that the bounds are tighter for larger x. When $a = 1$ and $a = 2$, the bounds provided by Theorem 2.11 are remarkably accurate across all values of x. The relative errors are consistently very low (on the order of 10^{-7} to 10^{-5}) for both $k = 1$ and $k = 2$. This suggests that Theorem 2.11 is particularly effective for integer and small fractional values of a. However, the lower bound for

Table 5
Comparisons for upper bound 2.4.

a	x	$\Gamma(a,x)$	Theorem 2.4	RE
0.6	0.1	1.085772151	1.666755811	0.535088011
	0.2	0.899439256	1.168935017	0.299626417
	0.3	0.763087263	0.916975188	0.201664912
	0.5	0.567529651	0.631833161	0.113304229
0.7	0.1	1.024394017	1.418519144	0.384739778
	0.2	0.870855068	1.061505117	0.218922821
	0.3	0.752330325	0.863771059	0.148127398
	0.5	0.574369634	0.622272361	0.083400522
0.8	0.1	0.974646826	1.213444408	0.245009347
	0.2	0.848080029	0.968251804	0.141698626
	0.3	0.745037651	0.816844704	0.096380436
	0.5	0.583055395	0.614753622	0.054365721
0.9	0.1	0.935162243	1.044195918	0.116593325
	0.2	0.830788149	0.887719986	0.068527503
	0.3	0.741193516	0.775916249	0.046847053
	0.5	0.593723165	0.609434499	0.026462390

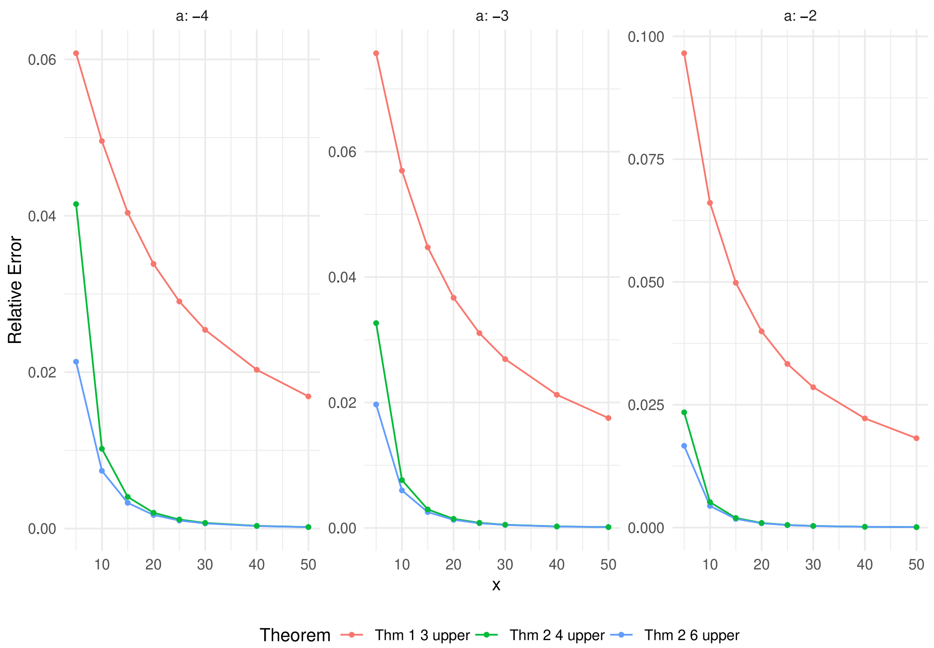

Fig. 3. Comparisons for Upper Bounds 1.3 [1], 2.4, and 2.6.

$k = 2$ is available, with a higher relative error of 35.41%. As x increases, both the availability and accuracy of the bounds improve. At $x = 10$, the relative errors decrease to 2.51% for $k = 1$ and 1.47% for $k = 2$, and they continue to decrease with larger x. Thus, Theorem 2.11 provides useful bounds for the incomplete gamma function, with accuracy generally improving for larger values of x and for certain values of a. The relative errors tend to decrease as x increases, indicating that the bounds become tighter. The theorem appears to be particularly effective for small and larger values of a, while for larger a and smaller x, the bounds may be less accurate or not applicable.

Table 7 compares the exact values of the incomplete gamma function, $\Gamma(a,x)$, and the approximations provided by Theorem 2.12 for different values of a and x, with k set to 1 (odd) and 2 (even). According to the given conditions, the approximation yields an upper bound when $a \geq 1$ and k is odd or when $0 < a < 1$ and k is even. Conversely, it provides a lower bound when $0 < a < 1$ and k is odd or when $a \geq 1$ and k is even. According to the table, for $a = 0.5$ (where $0 < a < 1$), the approximation with $k = 2$ (even) consistently overestimates $\Gamma(a,x)$, serving as an upper bound, while $k = 1$ (odd) underestimates it, acting as a lower bound, which aligns with the stated behavior. As x increases, the relative errors decrease, indicating improved accuracy of the approximations at larger x values. For $a = 1$, both $k = 1$ and $k = 2$ approximations closely match the exact values across all x, with negligible relative errors, suggesting high precision of the theorem at this specific value of a. When $a = 1.5$ (with $a > 1$), the approximation with $k = 1$ (odd) consistently overestimates $\Gamma(a,x)$, providing an upper bound, while $k = 2$ (even) underestimates it, providing a lower bound,

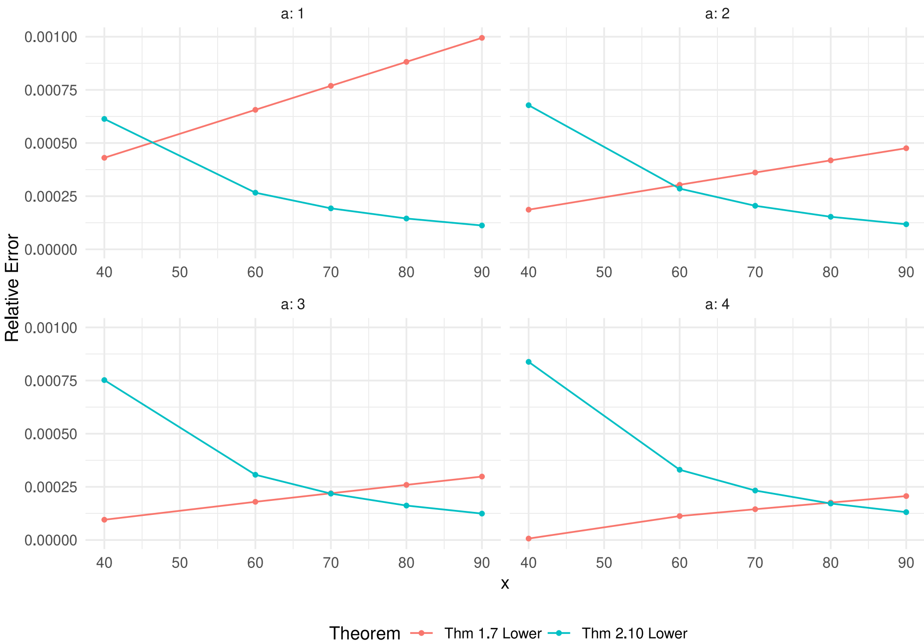

Fig. 4. Comparisons for Lower Bounds 1.7 [1] and 2.10.

Table 6
Comparisons for bounds 2.11.

a	x	$\Gamma(a,x)$	Theorem 2.11 ($k=1$, odd)	RE	Theorem 2.11 ($k=2$, even)	RE
0.5	5	2.774578×10^{-3}	2.782178×10^{-3}	2.739338×10^{-3}	2.770951×10^{-3}	1.307353×10^{-3}
	10	1.372612×10^{-5}	1.373277×10^{-5}	4.84532×10^{-4}	1.372457×10^{-5}	1.134131×10^{-4}
	15	7.657738×10^{-8}	7.659051×10^{-8}	1.714521×10^{-4}	7.657599×10^{-8}	1.809615×10^{-5}
	20	4.501324×10^{-10}	4.501702×10^{-10}	8.396816×10^{-5}	4.501329×10^{-10}	1.027637×10^{-6}
	25	2.725046×10^{-12}	2.725183×10^{-12}	5.027758×10^{-5}	2.725065×10^{-12}	6.829022×10^{-6}
	30	1.681284×10^{-14}	1.681342×10^{-14}	3.460803×10^{-5}	1.6813×10^{-14}	9.066783×10^{-6}
1	5	6.737946×10^{-3}	6.737947×10^{-3}	2.198106×10^{-7}	6.737947×10^{-3}	2.198106×10^{-7}
	10	4.539942×10^{-5}	4.539993×10^{-5}	1.127994×10^{-5}	4.539993×10^{-5}	1.127994×10^{-5}
	15	3.058989×10^{-7}	3.059023×10^{-7}	1.127994×10^{-5}	3.059023×10^{-7}	1.127994×10^{-5}
	20	2.06113×10^{-9}	2.061154×10^{-9}	1.127994×10^{-5}	2.061154×10^{-9}	1.127994×10^{-5}
	25	1.388779×10^{-11}	1.388794×10^{-11}	1.127994×10^{-5}	1.388794×10^{-11}	1.127994×10^{-5}
	30	9.357517×10^{-14}	9.357623×10^{-14}	1.127994×10^{-5}	9.357623×10^{-14}	1.127994×10^{-5}
2	5	4.042767×10^{-2}	4.042768×10^{-2}	2.15347×10^{-7}	4.042768×10^{-2}	2.15347×10^{-7}
	10	4.993991×10^{-4}	4.993992×10^{-4}	2.173759×10^{-7}	4.993992×10^{-4}	2.173759×10^{-7}
	15	4.894388×10^{-6}	4.894437×10^{-6}	9.99324×10^{-6}	4.894437×10^{-6}	9.99324×10^{-6}
	20	4.328378×10^{-8}	4.328423×10^{-8}	1.02996×10^{-5}	4.328423×10^{-8}	1.02996×10^{-5}
	25	3.610828×10^{-10}	3.610865×10^{-10}	1.048812×10^{-5}	3.610865×10^{-10}	1.048812×10^{-5}
	30	2.900832×10^{-12}	2.900863×10^{-12}	1.061584×10^{-5}	2.900863×10^{-12}	1.061584×10^{-5}
5	5	1.057184×10^{1}	NA	NA	6.828235×10^{0}	3.541109×10^{-1}
	10	7.020645×10^{-1}	7.19716×10^{-1}	2.514227×10^{-2}	6.917769×10^{-1}	1.465339×10^{-2}
	15	2.055939×10^{-2}	2.067453×10^{-2}	5.600448×10^{-3}	2.052263×10^{-2}	1.788019×10^{-3}
	20	4.06674×10^{-4}	4.075236×10^{-4}	2.089118×10^{-3}	4.064925×10^{-4}	4.463485×10^{-4}
	25	6.405801×10^{-6}	6.412191×10^{-6}	9.975747×10^{-4}	6.405182×10^{-6}	8.534963×10^{-5}
	30	1.156933×10^{-7}	1.157428×10^{-7}	7.082905×10^{-4}	1.15689×10^{-7}	1.228378×10^{-5}

again confirming the expected behavior. The relative errors decrease with increasing x, but remain slightly larger compared to the $a=1$ case. At higher values of a, such as $a=5$, the approximation with $k=1$ significantly overestimates $\Gamma(a,x)$ when x is small, reflected by large relative errors, but the accuracy improves as x increases.

4. Conclusions

In this paper, we introduced new bounds for the upper incomplete gamma function using various methods. We use the log-convexity of the function $H(a,x)=\frac{\Gamma(a,x)}{x^{a-1}e^{-x}}$ with respect to a, and established an interpolatory upper bound for $\Gamma(a,x)$ applicable to

Table 7
Comparisons for bounds 2.12.

a	x	$\Gamma(a,x)$	Theorem 2.12 ($k=1$, odd)	RE	Theorem 2.12 ($k=2$, even)	RE
0.5	5	2.774578×10^{-3}	2.739365×10^{-3}	1.269127×10^{-2}	2.812415×10^{-3}	1.363696×10^{-2}
	10	1.372612×10^{-5}	1.367307×10^{-5}	3.865485×10^{-3}	1.37501×10^{-5}	1.74654×10^{-3}
	15	7.657738×10^{-8}	7.643578×10^{-8}	1.849105×10^{-3}	7.661886×10^{-8}	5.416759×10^{-4}
	20	4.501324×10^{-10}	4.496468×10^{-10}	1.078922×10^{-3}	4.502404×10^{-10}	2.397859×10^{-4}
	25	2.725046×10^{-12}	2.723126×10^{-12}	7.044778×10^{-4}	2.7254×10^{-12}	1.30007×10^{-4}
	30	1.681284×10^{-14}	1.680453×10^{-14}	4.94511×10^{-4}	1.68142×10^{-14}	8.0744×10^{-5}
1	5	6.737946×10^{-3}	6.737947×10^{-3}	2.198106×10^{-7}	6.737947×10^{-3}	2.198106×10^{-7}
	10	4.539942×10^{-5}	4.539993×10^{-5}	1.127994×10^{-5}	4.539993×10^{-5}	1.127994×10^{-5}
	15	3.058989×10^{-7}	3.059023×10^{-7}	1.127994×10^{-5}	3.059023×10^{-7}	1.127994×10^{-5}
	20	2.06113×10^{-9}	2.061154×10^{-9}	1.127994×10^{-5}	2.061154×10^{-9}	1.127994×10^{-5}
	25	1.388779×10^{-11}	1.388794×10^{-11}	1.127994×10^{-5}	1.388794×10^{-11}	1.127994×10^{-5}
	30	9.357517×10^{-14}	9.357623×10^{-14}	1.127994×10^{-5}	9.357623×10^{-14}	1.127994×10^{-5}
1.5	5	1.64538×10^{-2}	1.674056×10^{-2}	1.742813×10^{-2}	1.60282×10^{-2}	2.586668×10^{-2}
	10	1.504286×10^{-4}	1.511234×10^{-4}	4.618193×10^{-3}	1.500702×10^{-4}	2.382631×10^{-3}
	15	1.22303×10^{-6}	1.225608×10^{-6}	2.107893×10^{-3}	1.222237×10^{-6}	6.489379×10^{-4}
	20	9.442724×10^{-9}	9.454112×10^{-9}	1.206049×10^{-3}	9.44028×10^{-9}	2.587737×10^{-4}
	25	7.080147×10^{-11}	7.085686×10^{-11}	7.822123×10^{-4}	7.079265×10^{-11}	1.247041×10^{-4}
	30	5.209389×10^{-13}	5.212252×10^{-13}	5.496602×10^{-4}	5.209042×10^{-13}	6.663158×10^{-5}
2	5	4.042767×10^{-2}	4.211217×10^{-2}	4.166689×10^{-2}	3.743304×10^{-2}	7.407387×10^{-2}
	10	4.993991×10^{-4}	5.044437×10^{-4}	1.010123×10^{-2}	4.965617×10^{-4}	5.681602×10^{-3}
	15	4.894388×10^{-6}	4.916287×10^{-6}	4.474324×10^{-3}	4.887197×10^{-6}	1.469311×10^{-3}
	20	4.328378×10^{-8}	4.339271×10^{-8}	2.516591×10^{-3}	4.325878×10^{-8}	5.775959×10^{-4}
	25	3.610828×10^{-10}	3.616652×10^{-10}	1.613069×10^{-3}	3.609815×10^{-10}	2.803394×10^{-4}
	30	2.900832×10^{-12}	2.90409×10^{-12}	1.122975×10^{-3}	2.900386×10^{-12}	1.539679×10^{-4}
5	5	1.057184×10^{1}	2.105608×10^{1}	9.917145×10^{-1}	NA	NA
	10	7.020645×10^{-1}	7.566655×10^{-1}	7.777203×10^{-2}	6.485704×10^{-1}	7.61954×10^{-2}
	15	2.055939×10^{-2}	2.111769×10^{-2}	2.715875×10^{-2}	2.007477×10^{-2}	3.391237×10^{-2}
	20	2.550507×10^{-4}	2.562046×10^{-4}	4.691118×10^{-2}	2.545045×10^{-4}	4.892121×10^{-2}
	25	3.240007×10^{-6}	3.241465×10^{-6}	6.667346×10^{-2}	3.23925×10^{-6}	6.788247×10^{-2}
	30	4.07616×10^{-8}	4.079023×10^{-8}	8.778166×10^{-2}	4.075017×10^{-8}	8.786413×10^{-2}

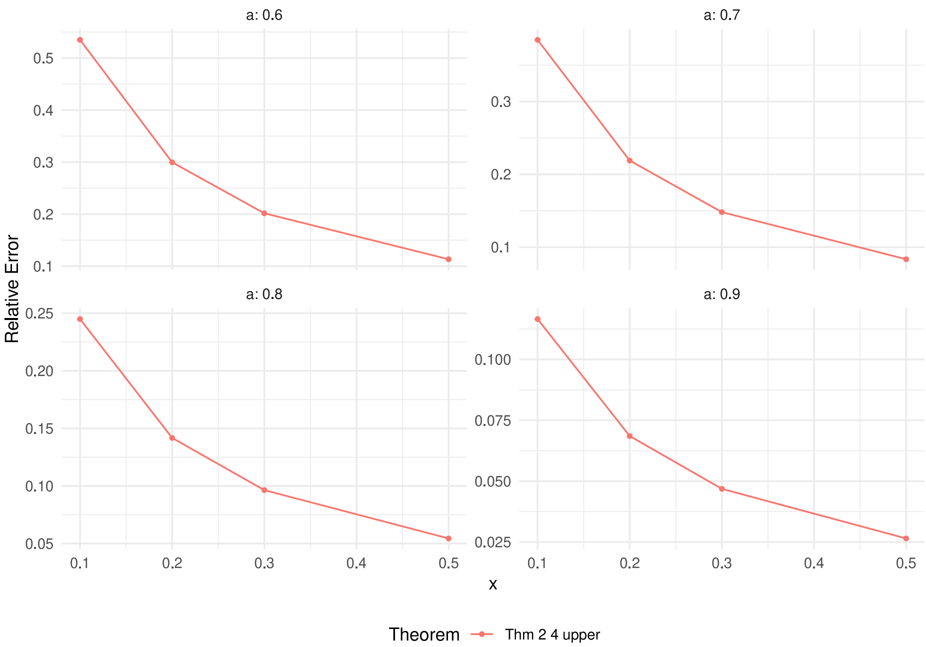

Fig. 5. Comparisons for Upper Bound 2.4 for various values of a.

all real $a \geq 1$ and $x > 0$. This approach allowed us to provide bounds that are particularly effective for intermediate values of x and provide improvements over some of the previously known bounds in the literature. Our empirical study shows that the new bounds in Theorems 2.4, 2.6, 2.8, and 2.10 consistently outperform existing bounds for the incomplete gamma function $\Gamma(a,x)$ across various conditions including both positive and negative values of a, as well as for large x. A drawback of the new bounds is that they are not as generally applicable as the bounds of Pinelis [1] nor do they have the correct asymptotic form as x approaches 0. However, these new theorems provide tighter upper and lower bounds with significantly lower relative errors compared to existing bounds from [1]. These findings suggest that the proposed new bounds offer more accurate approximations of the incomplete gamma function, which could have important implications for applications in probability theory, statistics, and other fields that require precise estimates of $\Gamma(a, x)$, especially when $x \gg a$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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