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ORIGINAL ARTICLE

Change Point Detection in Linear Failure Rate Distribution Under Random Censorship

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Abstract
In this paper, we develop a procedure for change point detection problem in the linear failure rate (LFR) distribution for random censored data. The asymptotic results of the associated test statistic have been established. Moreover, we construct the confidence sets for change locations based on the confidence distribution (CD). Simulations have been conducted to investigate the performance of the proposed procedure under different random censoring distributions. Two real data applications are provided to illustrate the detection procedure.

Keywords Change point · Linear failure rate · Confidence sets · Confidence distribution

1 Introduction

Change point analysis plays an important role in statistical analysis. The change point problem was initially introduced by [13, 14] to identify a single structural change in a parameter. Since then, it has been extensively studied in the literature. For example, [23, 24] studied the Bayesian approach to detect changes in the mean of a normal distribution. Chen and Gupta [4] and Cosörgö and Horváth [5] and established asymptotic properties on various parametric change point models. The likelihood ratio test for the known and unknown variance for testing the mean change was discussed by [7]. The asymptotic null distribution of a single change point detection for known and unknown variance was investigated by [32]. The multiple change point problem was first considered by [30] who proposed a binary segmentation (BS) procedure, which

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applies recursively to all subsequent partitions until no further significant change is detected. Chen and Gupta [3] studied the variance change point problem for univariate Gaussian model using the information criterion approach. The change point problem for generalized lambda distribution was studied by [11]. Ning et al. [10, 12] proposed nonparametric methods to detect the different types of changes in the mean. Ratnasingam and Ning [16] studied a confidence distribution based detection procedure for a skew normal change-point model incorporating modified information criterion. Recently, [18] proposed a method based on the modified information criterion and the confidence distribution for detecting and estimating changes in a three-parameter Weibull distribution.

Exponential, Rayleigh, linear failure rate, or generalized exponential distributions are often used in life-testing and reliability studies. The linear failure rate (LFR) distribution can be easily generalized to many well-known distributions such as the exponential distribution and the Rayleigh distribution. Bain [1] studied the statistical properties of the parameters of LFR distribution in the context of type II censorship. The probability density function of the LFR distribution is

$$f(x) = (\lambda + 2\beta x)e^{-(\lambda x + 2\beta x^2)}, \quad x > 0, \lambda > 0, \ \beta > 0, \tag{1}$$

where λ and β are the shape and scale parameters, respectively. The cumulative distribution function (CDF) of the LFR distribution is

$$F(x) = 1 - e^{-(\lambda x + \beta x^2)}. \tag{2}$$

The LFR hazard function is

$$h(x) = \frac{f(x)}{1 - F(x)} = (\lambda + 2\beta x), \tag{3}$$

where $x > 0, \lambda > 0, \beta > 0$. The LFR distribution is very useful in life testing and reliability studies for modeling the life length of a system or component when failures occur randomly and also from aging or wear-out. The following are some useful properties of the LFR distribution.

a. If $\lambda = 0$ and $\beta \neq 0$, then the LFR distribution is reduced to the Rayleigh distribution with parameter β.
b. If $\beta = 0$ and $\lambda \neq 0$, then the LFR distribution is reduced to the exponential distribution with parameter λ.

The LFR distribution with its properties and applications has been studied extensively in the literature. For example, the recurrence relations for the moment of the order statistics from the LFR distribution were established by [2]. In the life-testing and reliability studies, [22] studied various LFR distribution properties and applications. The Monte Carlo methods for Bayesian inference on the linear hazard distribution were developed by [9]. Sarhan and Kundu [19] considered the generalized linear failure rate distribution and discussed its properties and the maximum likelihood estimations (MLEs). Recently, [17] studied the statistical properties of a new four-parameter called as Lomax-linear failure rate distribution. To the best of our knowledge, a few works focus on the change point problem of the LFR distribution. In this paper, we propose a detection procedure regarding the LFR distribution based on the likelihood ratio test (LRT) under random censorship.

This paper is organized as follows. In Sect. 2, we develop a change point detection procedure for the LFR distribution and the procedure to construct the confidence set of a change with a given nominal level through the confidence distribution (CD). Simulations to investigate the performance of the proposed procedures in terms of powers, coverage probabilities, and lengths of confidence sets are conducted in Sect. 3. Two real data applications are given in Sect. 4 to illustrate the proposed procedures. Some discussion is provided in Sect. 5.

2 Main Results

2.1 Likelihood Ratio Test for LFR Distribution

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent observations belonging to a LFR distribution. The change point problem for a LFR distribution is defined as follows.

$$X_i \sim \begin{cases} \mathrm{LFR}(\lambda_L, \beta_L); & i = 1, \ldots, k \\ \mathrm{LFR}(\lambda_R, \beta_R); & i = (k+1), \ldots, n, \end{cases} \tag{4}$$

where the PDF and CDF of LFR distribution are given in (1) and (15), respectively. We are testing the following hypotheses.

$$H_0 : \lambda_1 = \lambda_2 = \cdots = \lambda_n = \lambda; \quad \beta_1 = \beta_2 = \cdots = \beta_n = \beta,$$

versus

$$H_1 : \underbrace{\lambda_1 = \cdots = \lambda_k}_{\lambda_L} \neq \underbrace{\lambda_{k+1} = \cdots = \lambda_n}_{\lambda_R}; \quad \underbrace{\beta_1 = \cdots = \beta_k}_{\beta_L} \neq \underbrace{\beta_{k+1} = \cdots = \beta_n}_{\beta_R},$$

where (λ, β), (λ_L, β_L) and (λ_R, β_R) are unknown parameters and need to be estimated. Further, k is the unknown change location and needs to be estimated as well. Let $1 < r < n$ be a natural number and is the number of observations presented if the data sample is censored. Suppose $X_1, \ldots, X_r$ denotes the r smallest observations from a sample of size n from the LFR model. Following [22], the likelihood function of the LFR distribution is,

$$L = \frac{n!}{(n-r)!} \prod_{i=1}^{r} (\lambda + 2\beta X_{(i)}) \exp\left(-(\lambda U_1 + \beta U_2)\right), \tag{5}$$

where

$$U_1 = \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)} \quad \text{and} \quad U_2 = \sum_{i=1}^{r} X_{(i)}^2 + (n-r)X_{(r)}^2. \tag{6}$$

Under the null hypothesis, the log-likelihood function is given as,

$$l_{H_0} = \log(L_0) = \log\left(\frac{n!}{(n-r)!}\right) + \sum_{i=1}^{r} \left\{\log\left(\lambda + 2\beta X_{(i)}\right) - (\lambda U_1 + \beta U_2)\right\}. \tag{7}$$

The first derivatives of the log likelihood function are given as below.

$$\frac{\partial l_{H_0}}{\partial \lambda} = \sum_{i=1}^{r} \frac{1}{(\lambda + 2\beta x_i)} - U_1, \quad \text{and} \quad \frac{\partial l_{H_0}}{\partial \beta} = \sum_{i=1}^{r} \frac{2x_i}{(\lambda + 2\beta x_i)} - U_2.$$

We note that the likelihood function in (5) is defined for censored data if $r < n$. If $n = r$, the complete sample is considered. Then, for a complete sample, U_1 and U_2 can be simplified as follows.

$$U_1 = \sum_{i=1}^{n} X_{(i)} \quad \text{and} \quad U_2 = \sum_{i=1}^{n} X_{(i)}^2,$$

According to [1], we have

$$\hat{\lambda} U_1 + \hat{\beta} U_2 = r. \tag{8}$$

where U_1 and U_2 are defined in (6), and $\hat{\lambda}$ and $\hat{\beta}$ are MLEs of λ and β. We can solve Eq. (8) for $\hat{\lambda}$ and consider $\hat{\lambda}$ as a function of $\hat{\beta}$, then we may substitute $\hat{\lambda} = \hat{\lambda}(\hat{\beta})$ into the likelihood function while treating the other occurrence of β as $\hat{\beta}$. Bain [1] proposed this idea to achieve a reduction of the MLE problem to a single-variable iterative procedure which reduces the two-variable iteration to that in a single variable say $\hat{\beta}$ through the so-called pseudo-likelihood function. Using the relation (8) and the likelihood function (5) we construct the pseudo-likelihood,

$$L = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left\{\left(\frac{r - \beta U_2}{U_1}\right) + 2\beta X_{(i)}\right\} \exp(-r). \tag{9}$$

Then, the pseudo log-likelihood function of the LFR distribution is

$$l_{H_0}(\lambda, \beta) = \sum_{i=1}^{r} (n-r+i) + \sum_{i=1}^{r} \log\left\{\left(\frac{r - \beta U_2}{U_1}\right) + 2\beta X_{(i)}\right\} - r. \tag{10}$$

Now the maximum likelihood estimate (MLE) of β is obtained by

$$\begin{aligned}\frac{\partial l_{H_0}(\lambda,\beta)}{\partial\beta} &= \frac{\partial}{\partial\beta}\left[\sum_{i=1}^{r}\log\left\{\left(\frac{r-\beta U_2}{U_1}\right)+2\beta X_{(i)}\right\}\right] \\ &= \sum_{i=1}^{r}\left[\left(\frac{U_1}{r-\beta U_2+2\beta X_{(i)}U_1}\right)\cdot\frac{\partial}{\partial\beta}\left(\frac{r}{U_1}+\frac{1}{U_1}\big(2\beta X_{(i)}U_1-\beta U_2\big)\right)\right] \\ &= \sum_{i=1}^{r}\left[\left(\frac{U_1}{r-\beta U_2+2\beta X_{(i)}U_1}\right)\cdot\frac{1}{U_1}\cdot\big(2X_{(i)}U_1-U_2\big)\right] \\ &= \sum_{i=1}^{r}\left[\frac{2U_1X_{(i)}-U_2}{r-\beta U_2+2\beta X_{(i)}U_1}\right]. \end{aligned} \tag{11}$$

Similar to [1, 22], we let $h(\beta)$ as

$$h(\beta)=\sum_{i=1}^{r}\left[\frac{2U_1X_{(i)}-U_2}{r-\beta U_2+2\beta X_{(i)}U_1}\right]. \tag{12}$$

Taking the first derivative of h gives

$$\begin{aligned}\frac{\partial h(\beta)}{\partial\beta} &= \sum_{i=1}^{r}\left[-\left(\frac{2U_1X_{(i)}-U_2}{\big(r-\beta U_2+2\beta X_{(i)}U_1\big)^2}\right)\cdot\big(-U_2^2+2U_1X_{(i)}\big)\right] \\ &= -\sum_{i=1}^{r}\left[\frac{\big(2U_1X_{(i)}-U_2\big)^2}{\big(r-\beta U_2+2\beta X_{(i)}U_1\big)^2}\right]<0. \end{aligned} \tag{13}$$

Therefore, the unique value of β such that $h(\beta)=0$ for $\hat{\beta}\in\big(0,\frac{r}{U_2}\big)$. The MLE of λ is

$$\hat{\lambda}=\frac{r-\hat{\beta}U_2}{U_1}. \tag{14}$$

Remark The "pseudo-likelihood" method proposed by [1] simplifies the estimation procedure; however, this method is easy to produce misleading results as pointed out by [22]. To tackle this issue, [1] suggested the range of searching $\hat{\beta}$ be restricted within $(0, r/U_2)$. Sen and Bhattacharyya [22] also verified that the solution found in this restricted range is indeed the solution of the original likelihood function, that is, maximizes the original likelihood function. This restricted range is satisfied due to Eq. (8).

Suppose $X_1,\dots,X_k$ be a sequence of independent random variables with the density function $f(x,\Theta_L)$ and $X_{k+1},\dots,X_n$ coming from the population with the density function $f(x, \Theta_R)$. Now the log-likelihood function is defined as

$$\ell(k, \Theta_L, \Theta_R) = \sum_{i=1}^{k} \log\big(f(x_i, \Theta_L)\big) + \sum_{i=k+1}^{n} \log\big(f(x_i, \Theta_R)\big), \tag{15}$$

where Θ_L and Θ_R are the parameter space of the pre-change and post-change distributions, respectively. Thus, under the alternative hypothesis, the log-likelihood function is,

$$l_{H_1} = \log(L_1) = l_{H_1}(k, \lambda_L, \beta_L, \lambda_R, \beta_R) = \sum_{i=1}^{k} \log(f(x_i, \lambda_L, \beta_L)) + \sum_{i=k+1}^{n} \log(f(x_i, \lambda_R, \beta_R)). \tag{16}$$

The MLEs of λ_L, β_L, λ_R, and β_R can be estimated by solving the following equations.

$$\frac{\partial l_{H_1}}{\partial \lambda_L} = 0, \quad \frac{\partial l_{H_1}}{\partial \beta_L} = 0, \quad \frac{\partial l_{H_1}}{\partial \lambda_R} = 0, \quad \text{and} \quad \frac{\partial l_{H_1}}{\partial \beta_R} = 0.$$

The log-likelihood ratio can be defined as,

$$\Lambda_k = -2\log\left(\frac{L_0}{L_1}\right) = -2\Big\{\log L_0 - \log L_1\Big\}. \tag{17}$$

The log-likelihood ratio test statistic Z_n can be obtained as $Z_n = \max_{1<k<n} \Lambda_k$. We have sufficient evidence to reject the null hypothesis for a large value of the log-likelihood ratio test statistic Z_n if there exists a change. If the change point location is in the beginning or the end of the data set, we may not have enough observations to estimate the MLEs of the parameters. Thus, we consider the trimmed version of the test statistic. Reprising [34], the trimmed version of the test statistic is defined as follows.

$$\widetilde{Z}_n = \max_{\ell<k<n-\ell} \Lambda_k, \tag{18}$$

where $\ell = 2[\log(n)]$. We can estimate the change point by $\hat{k}$ such that

$$\hat{k} = \max_{\ell<k<n-\ell} \Lambda_k. \tag{19}$$

This procedure can easily be replicated if a second change point is suspected.

Theorem 2.1 *Under the null hypothesis, as $n \longrightarrow \infty$, for all $t \in \mathbb{R}$, we have*

$$\lim_{n\longrightarrow\infty} P\left(a \log u(n)\, \widetilde{Z}_n \frac{1}{2} - b\log u(n) \le t\right) = e^{-e^t}, \tag{20}$$

where

$$a(\log u(n)) = (2\log\log u(n))^{1/2}$$
$$b(\log u(n)) = 2\log\log u(n) + \frac{3}{2}\log\log\log u(n) - \log\Gamma\left(\frac{3}{2}\right),$$
$$u(n) = \frac{n^2 - 2n\log(n) + \big(2\log(n)\big)^2}{(2\log n)^2}.$$

Proof Following [5] Theorem 1.3.1,

$$\begin{aligned}
1-\alpha &= P\left(\widetilde{Z}_n < C_n^\alpha \,\middle|\, H_0\right) \\
&= P\left(0 < \widetilde{Z}_n\frac{1}{2} < C_n^\alpha \,\middle|\, H_0\right) \\
&= P\bigg(-b\log u(n) < a\log(u(n))\widetilde{Z}_n\frac{1}{2} - b\log u(n) \\
&\quad < a\log u(n)(C_n^\alpha)^{1/2} - b\log u(n) \,\Big|\, H_0\bigg) \\
&= P\left(a\log u(n)\widetilde{Z}_n\frac{1}{2} - b\log u(n) < a\log u(n)(C_n^\alpha)^{1/2} - b\log u(n) \,\middle|\, H_0\right) \\
&\quad - P\left(a\log u(n)\widetilde{Z}_n\frac{1}{2} - b\log u(n) < -b\log u(n) \,\middle|\, H_0\right) \\
&\approx \exp\left\{-\exp\left\{b\log u(n) - a\log u(n)\big(C_n^\alpha\big)^{1/2}\right\}\right\} - \exp\left\{-\exp\left\{b\log u(n)\right\}\right\}.
\end{aligned}$$

Therefore,

$$C_n^\alpha \approx \left[\frac{\log\big(-\log\big(1-\alpha+\exp\{-\exp\{b\log u(n)\}\}\big)\big) - b\log u(n)}{-a\log u(n)}\right]^2. \quad (21)$$

□

The theoretical critical values for the change point problem can be approximated through Eq. (21). However, the theoretical critical values are unreliable when the sample size is considerably small. Thus, we compute the asymptotic critical values for the testing problem via simulations as follows. First, we generate data with various sample sizes $X_1, X_2, \ldots, X_n$ from LFRD(1, 1). The censoring time observations $Y_1, Y_2, \ldots, Y_n$ are from $Uniform(0, \theta_1)$ and $Exp(\theta_2)$, where θ_1 and θ_2 are the censoring parameters which determine the censoring proportion of $X_1, X_2, \ldots, X_n$. Three different values of $\theta = \{\theta_1, \theta_2\}$ to achieve 10, 20, and 30% censoring proportion of $X_1, X_2, \ldots, X_n$ are calculated by $P(X \geq Y) = 10\%$, $P(X \geq Y) = 20\%$, and $P(X \geq Y) = 30\%$, respectively. For example, when the censoring proportion is 10%, the censoring parameter θ can be obtained in the following way.

$$P(\delta = 1|\lambda, \beta, \theta) = P(Y \le X \le \infty, 0 \le Y \le \theta)$$
$$0.1 = \frac{1}{\theta}\int_0^{\theta} \left(1 - e^{-(\lambda x + \beta x^2)}\right) \mathrm{d}x.$$

The censoring parameter θ is selected for 10, 20, and 30% are 0.188, 0.376, and 0.565, respectively. We provide steps to obtain critical values via simulation as follows.

- Step 1: We generate data with various sample sizes $X_1, X_2, \ldots, X_n$ from LFRD(1, 1). The censoring time observations $Y_1, Y_2, \ldots, Y_n$ generated from $Uniform(0, \theta)$ or $Exp(\theta)$, where θ is the censoring parameter and it takes values 0.188, 0.376, and 0.565 for the desired nominal censoring proportions of 10, 20, and 30%, respectively.
- Step 2: For each generated sample, we calculate the log-likelihood ratio test statistic $\widetilde{Z}_n$.
- Step 3: We repeat the above steps $N = 1000$ times. Then the critical value is the $100(1-\alpha)$th quantile of the asymptotic distribution obtained in Step 2, where the significance level $\alpha = 0.01, 0.05$, and 0.1.

The asymptotic critical values are presented in Table 1.

2.2 Confidence Distribution, Profile Log-Likelihood and Deviance Function

Confidence distributions (CD) are distribution estimates to be interpreted as distributions of epistemic probabilities. The concept of a CD is analogous to a point estimator which may be considered a sample-dependent distribution that describes confidence intervals of all levels for a parameter of interest. Schweder and Hjort [20] provides a formal definition of the CD. In addition, the theoretical properties of the CD were extensively investigated by [21]. A detailed study of recent developments in CD has been given by [33]. More applications of the CD, such as bootstrap distributions, p-value functions, normalized likelihood functions, and Bayesian posteriors, among others, can be found in the literature. Interested readers may refer to [20, 25–28].

Cunen et al. [6] investigated the CD for change point analysis and construct confidence curves for change locations using the log-likelihood approach. Ratnasingam and Ning [16] examined the change point detection procedure based on the CD combined with the modified information criterion (MIC) to construct the confidence set for the change estimate for a skew normal change point model. In this paper, we study the CD-based procedure along with log-likelihood for LFR distribution with predefined censoring rates. Next, we define a procedure to construct a confidence curve for the LFR change point model.

By maximizing the log-likelihood function defined in (15) of a given k, we can obtain the profile log-likelihood function as follows.

$$\ell_{\text{prof}}(k) = \max_{\Theta_L, \Theta_R} \left(\ell(k, \Theta_L, \Theta_R)\right) = \ell\big(k, \hat{\Theta}_L, \hat{\Theta}_R\big), \tag{22}$$

Table 1 Critical values

Censoring proportion	n	α					
		0.10		0.05		0.01	
		Unif	Exp	Unif	Exp	Unif	Exp
10%	50	5.0479	5.7437	5.6971	6.4269	6.9139	7.8907
	70	5.2766	6.1986	5.9888	6.8573	7.1433	8.4002
	80	5.5543	6.2500	6.1159	7.0724	7.2960	9.4135
	100	5.6800	6.4602	6.3707	7.0710	8.6988	8.6506
	120	5.9543	6.8043	6.5633	7.5445	8.0500	8.5234
	150	6.1471	7.1785	6.8932	7.8760	8.7705	8.8895
	180	6.5444	7.1809	7.2683	7.8028	9.2532	9.4725
	200	6.7155	7.6634	7.4235	8.4876	9.2955	10.1574
	250	7.0765	7.7644	7.7020	8.6036	9.5498	10.8396
	300	7.2570	7.8172	8.1147	8.5838	9.6960	9.6316
20%	50	5.1018	5.8791	5.5282	6.4193	6.7381	8.1416
	70	5.3899	6.2923	6.1077	7.0305	7.2452	8.7102
	80	5.6840	6.5105	6.4724	7.1967	7.8984	9.0336
	100	5.7330	6.8311	6.4448	7.6074	8.0667	9.4676
	120	6.2675	6.9218	6.9924	7.6315	8.9820	8.7816
	150	6.4339	7.1589	7.1341	7.8910	8.6351	9.3840
	180	6.5514	7.4047	7.2280	8.2279	8.8312	9.8598
	200	6.6457	7.5086	7.3258	8.3124	8.8651	9.8869
	250	6.7988	7.6448	7.4857	8.1717	9.1376	9.7778
	300	7.0682	8.0576	7.7797	8.9030	9.1660	10.4283
30%	50	5.1473	5.9394	5.7756	6.5365	7.1700	8.3375
	70	5.5544	6.2517	6.0702	7.0560	7.7614	9.2413
	80	5.7875	6.6184	6.6001	7.5314	7.8760	9.4933
	100	6.0460	6.5152	6.7817	7.3224	8.2742	9.1177
	120	6.2877	6.8603	6.9330	7.7574	8.8020	9.6408
	150	6.4321	7.1060	7.2360	7.7950	8.8030	9.1790
	180	6.5584	7.4027	7.1352	8.1393	8.8008	9.1584
	200	6.6701	7.6056	7.5470	8.2645	9.2626	9.7597
	250	6.9870	7.7921	7.6455	8.6692	8.5804	10.6687
	300	7.1211	7.9195	7.8438	8.6550	9.8231	10.2005

where $\hat{\Theta}_L$ and $\hat{\Theta}_R$ are MLEs of Θ_L and Θ_R, respectively. The estimated change point location $\hat{k}$ corresponds to the $\max_k \left(\ell_{\text{prof}}(k)\right)$. The deviance function is defined as

$$\mathcal{D}(k, \mathbf{x}) = 2\left[\ell_{\text{prof}}(\hat{k}) - \ell_{\text{prof}}(k)\right], \tag{23}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The confidence curve for k based on the deviance function can be obtained through simulation.

$$cc(k, \mathbf{x}_{\text{obs}}) = \varphi_k\big(\mathcal{D}(k, \mathbf{x}_{\text{obs}})\big) = P_{k,\hat{\Theta}_L,\hat{\Theta}_R}\big(\mathcal{D}(k, \mathbf{x}) < \mathcal{D}(k, \mathbf{x}_{\text{obs}})\big). \tag{24}$$

where the $cc(k, \mathbf{x}_{\text{obs}}) < \alpha$ under the true value of k. By simulation, we compute

$$cc(k, \mathbf{x}_{\text{obs}}) = \frac{1}{B}\sum_{j=1}^{B} I\big(D(k, \mathbf{x}_j^*) < D(k, \mathbf{x}_{\text{obs}})\big), \tag{25}$$

for a large number of B of simulated copies of dataset $\mathbf{x}_{\text{obs}}$. For each possible value of k, we simulated data $\mathbf{x}_j^*$, $j = 1, \ldots, B$ from $f(\mathbf{x}, \Theta_L)$ and $f(\mathbf{x}, \Theta_R)$ to the left and right side of k, respectively. Furthermore, the change point location is estimated by (19). For more details, we refer the readers to [6, 16].

3 Simulation Study

In this section, we conduct a simulation study to evaluate the performance of the proposed method. First, we verify the null asymptotic distribution of $\widetilde{Z}_n$ stated in Theorem 2.1 numerically. The data is obtained from LFRD (1, 1). For different sample sizes $n = \{50, 100, 200\}$ and various censoring proportions, we sketch the standard Gumbel distribution quantile–quantile (Q–Q) plot for $\widetilde{Z}_n$ values in Figs. 1, 2 and 3. According to the graphs, we observe that the null asymptotic distribution of $\widetilde{Z}_n$ can be approximated to the standard Gumbel distribution and fit reasonably well when the sample size increases. This confirms the result given in Theorem 2.1.

Secondly, we conduct the power simulation study. In this case, simultaneous changes in the parameters are considered. The pre-change data are always generated from LFRD (1, 1). The post-change data are generated from LFRD (1.25, 1.25), LFRD (1.5, 1.5), LFRD (1.75, 1.75), and LFRD (2, 3). Simulations are conducted for sample sizes $n = 50$, 100, and 150. The values of true change point k are chosen to be {15, 20, 25}, {25, 35, 50}, {25, 50, 75} for sample sizes 50, 100, and 150, respectively. In our simulations, we consider only the true change-point positions below or equal to the midpoint of the data set due to the symmetric property of the performance. We generate the censoring time with different censoring times from a uniform distribution and an exponential distribution with different censoring rates considering 10, 20, and 30%. The results are based on 1000 replications. The results are summarized in Tables 2, 3 and 4. We notice that when the difference between the parameters increases the test power increases. For example, in Table 2 for sample size $n = 50, \alpha = 0.05, k = 15, \lambda = 1.25$, and $\beta = 1.25$ the power is 0.105, however, the power is 0.526 when $\lambda = 1.5$, and $\beta = 2.5$. Based on Tables 2, 3 and 4, it is evident that the power of the test increases as the sample size n increases. From the simulation results, we observe that the power of the test increases as the increase of censoring proportion when the censoring distribution is Exponential. For instance, when the sample size $n = 50$, $k = 15$, and the censoring proportion is 10%, the power

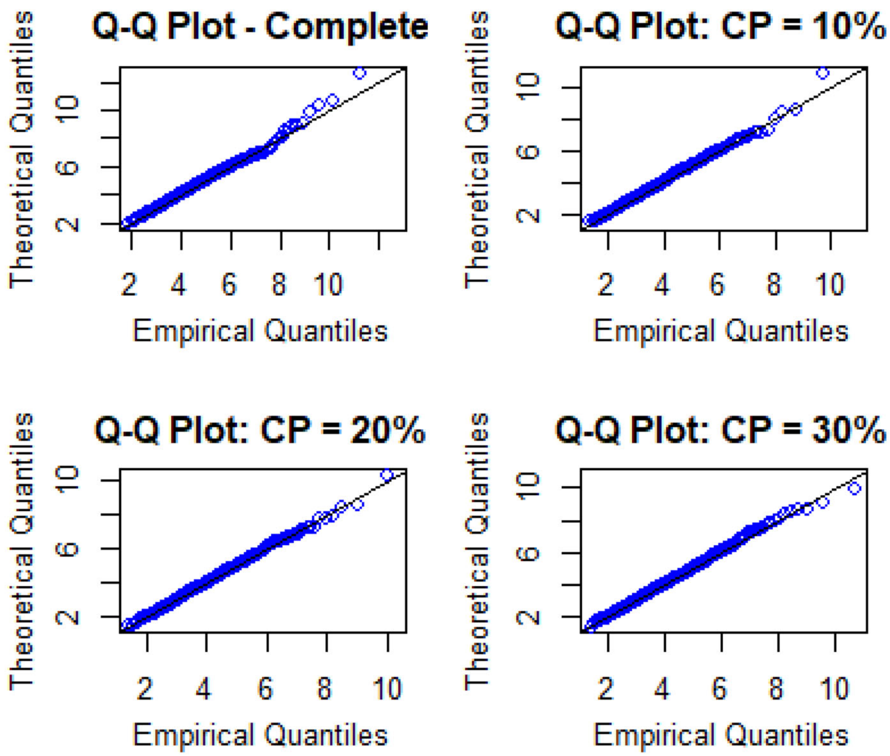

Fig. 1 The Gumbel Q–Q plot of $\widetilde{Z}_n$ for $n = 50$ for complete, and various censoring proportions, 10, 20, and 30%

of the test is 0.842. The power increases to 0.845 and 0.856 as a censoring proportion increases to 20 and 30%, respectively.

Next, we conduct a simulation study for the coverage probability and confidence sets of the change point estimator. We consider various sample sizes, including $n = \{50, 100, 150\}$, and the nominal level $\alpha = \{0.90, 0.95, 0.99\}$. Under the null hypothesis, the data are generated from LFRD (1,1). The post-change data are generated from LFRD (1.25, 1.25), LFRD (1.5, 1.5), LFRD (1.75, 1.75), and LFRD (2, 3). We generate the censoring time with different censoring times from an exponential distribution with different censoring rates, considering 10, 20, and 30%. We consider two criteria to determine the goodness of the procedure. They are the coverage probability and average size of confidence sets where the size of a confidence set is defined by the number of estimated k belonging to the confidence set for a given nominal level α. In general, if the procedure is good, then it should lead to a narrower confidence set $\{k : cc(k, x) \leq \alpha\}$ and the coverage probability preferably close to the nominal level α. The results are summarized in Tables 5 and 6. For example, in Table 5 for sample size $n = 50$, $k = 15$, $\lambda = 1.25$, $\beta = 1.25$, $\alpha = 0.50$, and censoring proportion is 10% the coverage probability is 0.39, and the average size of the confidence set is 17.30. As the differences between λ and β increase, we see a corresponding increase in the

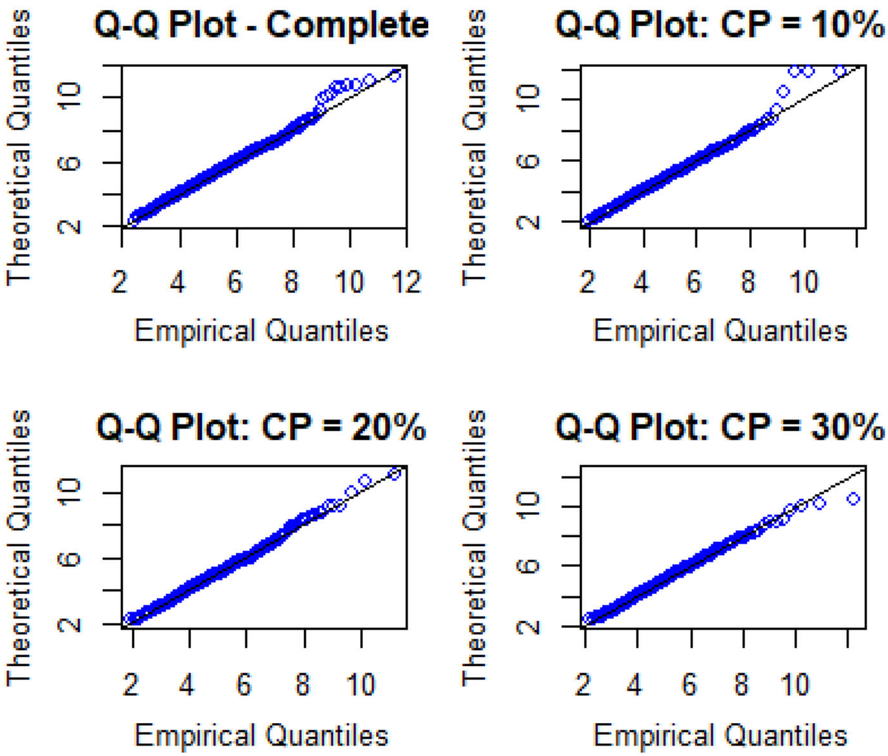

Fig. 2 The Gumbel Q–Q plot of $\widetilde{Z}_n$ for $n = 100$ for complete, and various censoring proportions, 10, 20, and 30%

coverage probability (CP) and decrease in the average size of confidence sets. We also observe that the censoring proportion increases the coverage probability decreases (Table 7).

4 Applications

In this section, we apply the proposed testing procedure to two real data sets. The first data set is from the R package *survival*: Veterans administration lung cancer data and the second data set is from the R package *relsurv*: acute myocardial infarction data set. As the censoring distribution, the uniform distribution is used in both applications.

4.1 Veterans' Administration Lung Cancer Study

We adopt the veterans' administration lung cancer data in the survival package. The data were initially studied by [8]. Survival times are classified into two categories based on the two treatment regimens for lung cancer and the patient's age survival times in two groups. Group 1 consists of 69 patients, while Group 2 consists of 68

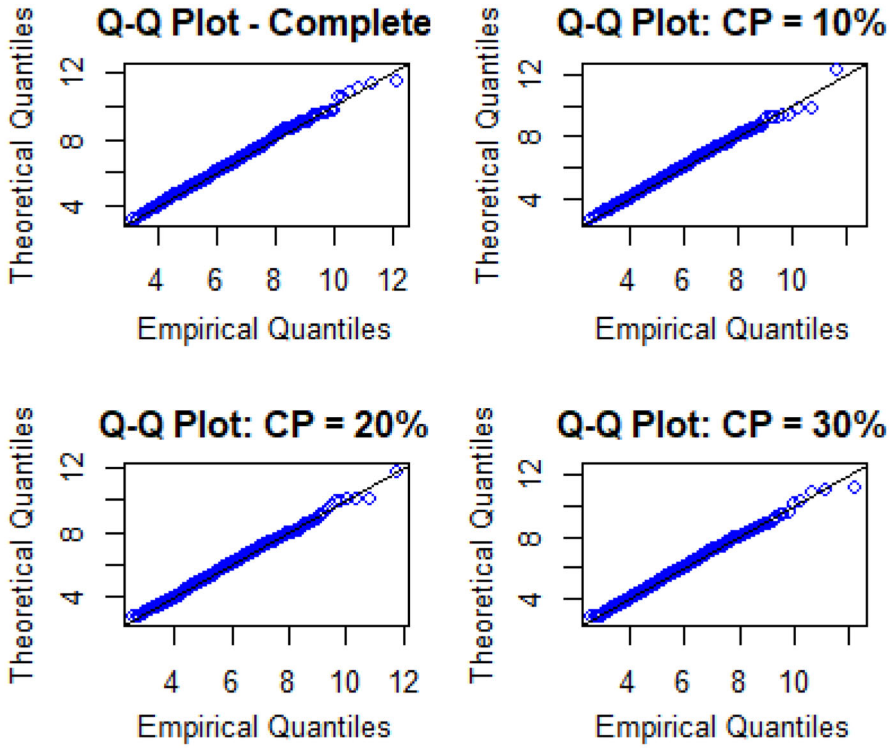

Fig. 3 The Gumbel Q-Q plot of $\widetilde{Z}_n$ for $n = 200$ for complete, and various censoring proportions, 10, 20, and 30%

Table 2 Power comparison for sample size $n = 50$ and $k = \{15, 20, 25\}$

Censoring Proportion	k	LFRD (λ, β)							
		(1.25, 1.25)		(1.5, 2.5)		(1.75, 2.75)		(2, 3)	
		Unif	Exp	Unif	Exp	Unif	Exp	Unif	Exp
10%	15	0.677	0.842	0.705	0.989	0.714	0.998	0.754	1.000
	20	0.683	0.849	0.719	0.988	0.725	0.999	0.768	1.000
	25	0.716	0.853	0.792	0.991	0.780	1.000	0.750	1.000
20%	15	0.770	0.845	0.790	0.986	0.852	0.998	0.860	1.000
	20	0.778	0.856	0.799	0.988	0.859	0.999	0.886	1.000
	25	0.789	0.872	0.805	0.992	0.865	1.000	0.895	1.000
30%	15	0.775	0.856	0.833	0.988	0.877	0.997	0.909	1.000
	20	0.792	0.871	0.849	0.989	0.872	0.999	0.918	1.000
	25	0.812	0.861	0.869	0.994	0.902	1.000	0.949	1.000

Table 3 Power comparison for sample size $n = 100$ and $k = \{25, 35, 50\}$

Censoring proportion	k	LFRD (λ, β)							
		(1.25, 1.25)		(1.5, 2.5)		(1.75, 2.75)		(2, 3)	
		Unif	Exp	Unif	Exp	Unif	Exp	Unif	Exp
10%	25	0.812	0.999	0.844	1.000	0.845	1.000	0.889	1.000
	35	0.835	1.000	0.861	1.000	0.882	1.000	0.905	1.000
	50	0.851	1.000	0.878	1.000	0.881	1.000	0.911	1.000
20%	25	0.849	1.000	0.901	1.000	0.935	1.000	0.961	1.000
	35	0.858	1.000	0.924	1.000	0.950	1.000	0.980	1.000
	50	0.876	1.000	0.947	1.000	0.967	1.000	0.985	1.000
30%	25	0.844	1.000	0.916	1.000	0.971	1.000	0.990	1.000
	35	0.855	1.000	0.930	1.000	0.977	1.000	0.990	1.000
	50	0.880	1.000	0.953	1.000	0.990	1.000	0.998	1.000

Table 4 Power comparison for sample size $n = 150$ and $k = \{25, 50, 75\}$

Censoring proportion	k	LFRD (λ, β)							
		(1.25, 1.25)		(1.5, 2.5)		(1.75, 2.75)		(2, 3)	
		Unif	Exp	Unif	Exp	Unif	Exp	Unif	Exp
10%	25	0.860	1.000	0.897	1.000	0.908	1.000	0.921	1.000
	50	0.872	1.000	0.916	1.000	0.941	1.000	0.958	1.000
	75	0.881	1.000	0.942	1.000	0.964	1.000	0.972	1.000
20%	25	0.874	1.000	0.901	1.000	0.943	1.000	0.970	1.000
	50	0.882	1.000	0.946	1.000	0.982	1.000	0.994	1.000
	75	0.893	1.000	0.957	1.000	0.988	1.000	0.999	1.000
30%	25	0.906	1.000	0.953	1.000	0.977	1.000	0.995	1.000
	50	0.919	1.000	0.976	1.000	0.999	1.000	1.000	1.000
	75	0.938	1.000	0.991	1.000	0.999	1.000	1.000	1.000

patients. We considered the censoring proportions 8 and 6%, respectively. Two groups are shown in Fig. 4.

We apply the proposed approach along with the binary segmentation procedure (see, [30]) to detect multiple change points. We found that Group 1 and Group 2 have a change point at 54th and 15th positions, respectively. Moreover, the 95% confidence sets for the estimated change location for Group 1 is $\{41, 42, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56\}$, and for Group 2 is $\{13, 14, 15\}$. The corresponding change point location and the 95% confidence sets are graphed in Figs. 5 and 6, respectively.

4.2 Acute Myocardial Infarction Data

The second data set is about acute myocardial infarction. For the purpose of this analysis, we consider a subset of data from a study carried out at the University

Table 5 Coverage probability and average size of confidence sets for $n = 50$ and $k = \{15, 20, 25\}$

Censoring proportion	k	α	LFRD (λ, β)							
			(1.25, 1.25)		(1.5, 2.5)		(1.75, 2.75)		(2, 3)	
			CP	Size	CP	Size	CP	Size	CP	Size
10%	15	0.50	0.39	17.30	0.54	17.08	0.56	16.24	0.61	15.91
		0.90	0.82	35.57	0.87	33.65	0.89	32.00	0.90	30.32
		0.95	0.89	39.05	0.92	37.31	0.93	35.90	0.94	34.48
		0.99	0.94	41.43	0.96	40.95	0.96	40.35	0.97	39.72
	20	0.50	0.40	17.56	0.55	17.14	0.58	16.55	0.62	15.71
		0.90	0.83	35.72	0.88	33.72	0.90	32.13	0.91	29.95
		0.95	0.90	39.06	0.93	37.39	0.95	36.02	0.95	34.05
		0.99	0.94	41.47	0.96	41.31	0.97	40.85	0.98	39.78
	25	0.50	0.41	17.59	0.56	17.04	0.59	16.68	0.63	15.72
		0.90	0.84	35.64	0.89	33.86	0.91	32.09	0.92	29.91
		0.95	0.91	39.04	0.94	37.52	0.95	36.04	0.95	34.16
		0.99	0.94	41.53	0.96	40.40	0.97	39.89	0.98	38.99
20%	15	0.50	0.37	16.58	0.49	16.39	0.55	15.79	0.59	15.01
		0.90	0.81	35.05	0.86	34.03	0.87	32.63	0.88	31.12
		0.95	0.88	38.28	0.92	37.55	0.92	36.19	0.92	35.02
		0.99	0.93	40.81	0.95	40.02	0.95	39.80	0.96	39.42
	20	0.50	0.38	16.53	0.50	16.27	0.56	15.93	0.60	15.32
		0.90	0.82	35.07	0.86	33.73	0.88	32.51	0.89	30.59
		0.95	0.89	38.30	0.92	37.59	0.93	36.36	0.94	34.57
		0.99	0.94	40.83	0.96	40.21	0.96	39.75	0.97	38.87
	25	0.50	0.39	16.72	0.51	16.39	0.57	15.93	0.60	15.23
		0.90	0.82	35.16	0.87	33.83	0.89	32.73	0.91	31.32
		0.95	0.88	38.41	0.92	37.58	0.94	36.58	0.95	35.34
		0.99	0.93	40.96	0.96	39.21	0.97	39.03	0.97	38.52
30%	15	0.50	0.37	16.72	0.48	16.30	0.51	15.73	0.56	15.21
		0.90	0.81	35.05	0.85	33.94	0.86	16.73	0.87	31.83
		0.95	0.88	38.18	0.90	37.41	0.91	36.61	0.92	35.63
		0.99	0.93	40.18	0.94	39.83	0.95	39.54	0.95	39.16
	20	0.50	0.38	16.80	0.49	16.44	0.53	15.79	0.57	15.09
		0.90	0.82	35.02	0.85	33.87	0.87	32.85	0.88	31.09
		0.95	0.88	38.24	0.90	33.87	0.91	36.60	0.92	35.19
		0.99	0.93	40.85	0.94	40.24	0.95	39.74	0.96	39.29
	25	0.50	0.40	16.71	0.50	16.31	0.55	15.60	0.58	15.00
		0.90	0.83	35.06	0.85	33.82	0.87	33.07	0.90	32.16
		0.95	0.89	38.25	0.91	37.53	0.93	36.86	0.94	36.03
		0.99	0.94	40.85	0.95	40.37	0.96	39.97	0.97	39.33

Table 6 Coverage probability and average size of confidence sets for $n = 100$ and $k = \{25, 35, 50\}$

Censoring proportion	k	α	LFRD (λ, β)							
			(1.25, 1.25)		(1.5, 2.5)		(1.75, 2.75)		(2, 3)	
			CP	Size	CP	Size	CP	Size	CP	Size
10%	25	0.50	0.33	26.34	0.56	24.48	0.60	21.19	0.64	19.29
		0.90	0.77	67.93	0.87	56.69	0.89	48.38	0.91	41.68
		0.95	0.85	77.46	0.92	67.23	0.93	58.98	0.95	51.29
		0.99	0.94	87.78	0.96	81.61	0.97	75.40	0.98	68.90
	35	0.50	0.34	26.28	0.57	22.48	0.62	19.62	0.65	17.27
		0.90	0.78	67.78	0.88	53.35	0.90	44.74	0.92	37.76
		0.95	0.87	77.62	0.93	64.72	0.94	55.24	0.95	46.72
		0.99	0.95	88.13	0.97	80.88	0.98	73.48	0.98	64.82
	50	0.50	0.36	25.81	0.58	23.59	0.63	19.95	0.66	17.36
		0.90	0.80	67.88	0.89	53.29	0.91	44.84	0.92	37.40
		0.95	0.89	78.11	0.93	63.92	0.95	55.21	0.96	46.53
		0.99	0.96	88.44	0.97	80.24	0.98	72.70	0.99	64.44
20%	25	0.50	0.30	26.56	0.51	25.65	0.58	22.43	0.61	21.07
		0.90	0.78	68.95	0.85	59.34	0.88	52.19	0.91	46.00
		0.95	0.87	79.11	0.91	69.87	0.93	62.61	0.94	56.34
		0.99	0.94	88.56	0.95	82.59	0.96	78.02	0.98	73.31
	35	0.50	0.33	27.24	0.52	24.01	0.59	21.42	0.62	18.86
		0.90	0.79	68.65	0.86	57.16	0.89	48.91	0.89	41.33
		0.95	0.88	78.85	0.93	68.04	0.94	59.55	0.94	50.79
		0.99	0.95	88.75	0.97	82.34	0.97	76.63	0.98	69.76
	50	0.50	0.33	26.15	0.54	24.59	0.61	22.63	0.64	18.71
		0.90	0.80	68.33	0.87	56.84	0.90	49.85	0.91	41.51
		0.95	0.89	78.95	0.92	67.64	0.93	60.13	0.95	51.14
		0.99	0.95	88.96	0.97	82.78	0.98	77.09	0.98	69.62
30%	25	0.50	0.30	25.43	0.49	26.62	0.55	24.36	0.60	23.07
		0.90	0.77	68.80	0.85	62.54	0.87	56.39	0.90	50.77
		0.95	0.85	78.61	0.91	72.52	0.93	67.22	0.94	60.83
		0.99	0.93	87.53	0.95	84.44	0.97	81.47	0.97	77.00
	35	0.50	0.31	26.20	0.50	25.32	0.56	23.34	0.61	20.84
		0.90	0.78	68.67	0.85	60.16	0.88	23.34	0.89	46.10
		0.95	0.86	78.84	0.92	70.83	0.94	64.10	0.95	56.43
		0.99	0.94	87.82	0.96	84.13	0.97	79.68	0.98	74.27
	50	0.50	0.33	26.03	0.51	25.26	0.57	22.55	0.63	20.42
		0.90	0.80	68.12	0.86	59.37	0.88	52.59	0.91	45.58
		0.95	0.87	78.27	0.92	70.30	0.93	63.51	0.95	56.08
		0.99	0.94	87.56	0.96	83.90	0.97	79.63	0.98	74.33

Table 7 Coverage probability and average size of confidence sets for $n = 150$ and $k = \{25, 50, 75\}$

Censoring proportion	k	α	LFRD (λ, β)							
			(1.25, 1.25)		(1.5, 2.5)		(1.75, 2.75)		(2, 3)	
			CP	Size	CP	Size	CP	Size	CP	Size
10%	25	0.50	0.26	32.13	0.52	28.37	0.60	25.36	0.63	21.79
		0.90	0.73	97.38	0.86	74.66	0.89	62.71	0.90	50.41
		0.95	0.83	114.21	0.91	91.29	0.94	77.74	0.95	63.56
		0.99	0.93	132.42	0.96	116.85	0.98	105.24	0.98	90.91
	50	0.50	0.27	37.16	0.54	25.62	0.63	22.06	0.67	18.25
		0.90	0.74	122.76	0.87	63.47	0.90	49.68	0.92	38.67
		0.95	0.85	150.45	0.93	79.05	0.95	62.06	0.95	48.80
		0.99	0.95	179.30	0.98	107.50	0.98	88.66	0.99	71.85
	75	0.50	0.28	32.16	0.57	25.39	0.64	20.72	0.68	17.77
		0.90	0.75	94.35	0.88	60.76	0.90	46.40	0.91	36.33
		0.95	0.86	111.96	0.93	76.63	0.95	58.90	0.95	45.88
		0.99	0.96	131.83	0.98	105.99	0.99	86.39	0.99	68.70
20%	25	0.50	0.25	32.25	0.47	28.57	0.56	25.81	0.60	23.58
		0.90	0.72	97.19	0.84	79.24	0.87	67.38	0.90	56.82
		0.95	0.83	114.68	0.90	96.72	0.93	84.50	0.95	72.47
		0.99	0.92	132.13	0.95	121.20	0.97	111.65	0.97	100.34

Table 7 continued

Censoring proportion	k	α	LFRD (λ, β)							
			(1.25, 1.25)		(1.5, 2.5)		(1.75, 2.75)		(2, 3)	
			CP	Size	CP	Size	CP	Size	CP	Size
	50	0.50	0.26	37.33	0.53	28.41	0.60	23.67	0.63	19.77
		0.90	0.74	124.25	0.86	70.23	0.89	55.38	0.91	43.70
		0.95	0.84	151.12	0.92	86.77	0.94	69.79	0.95	55.53
		0.99	0.95	180.05	0.97	114.36	0.98	98.04	0.98	81.35
	75	0.50	0.28	33.79	0.57	28.10	0.61	23.11	0.65	19.77
		0.90	0.75	96.18	0.87	68.78	0.89	53.40	0.92	42.18
		0.95	0.85	113.36	0.93	85.31	0.94	67.47	0.96	53.594
		0.99	0.96	132.58	0.97	113.42	0.98	95.75	0.99	79.66
30%	25	0.50	0.24	32.76	0.47	30.71	0.52	28.02	0.58	25.24
		0.90	0.72	98.02	0.81	83.83	0.85	73.13	0.89	62.17
		0.95	0.83	115.72	0.88	100.50	0.91	90.32	0.93	78.17
		0.99	0.93	132.73	0.95	123.86	0.96	115.86	0.97	106.38
	50	0.50	0.25	37.94	0.49	30.59	0.56	25.40	0.60	21.34
		0.90	0.73	124.51	0.85	76.70	0.87	61.82	0.89	49.51
		0.95	0.84	151.34	0.90	93.40	0.92	76.86	0.94	63.04
		0.99	0.94	180.20	0.96	120.20	0.97	105.72	0.98	90.80
	75	0.50	0.26	33.59	0.50	29.99	0.59	25.80	0.61	21.70
		0.90	0.75	97.45	0.85	74.48	0.88	60.55	0.90	49.11
		0.95	0.86	114.86	0.91	91.50	0.93	75.76	0.95	61.94
		0.99	0.95	133.31	0.97	118.68	0.97	104.43	0.98	89.79

Clinical Center in Ljubljana. The data were originally studied by [15]. The data set provides details of 972 patients between the ages of 40 and 80, of which 48% have censored outcomes. This data set was studied in the literature, for example, see [31] and [29]. Our proposed approach is used to identify structural changes in the data set. The binary segmentation method by [30] is used to identify potential multiple changes. The change point estimates are $\hat{k} = \{78, 184, 532, 842, 881, 962\}$. The 95% confidence sets of the change point estimates are indicated by the horizontal red dashed line in Fig. 7.

In comparison with [31], we obtain the following equation:

$$\begin{aligned}\hat{y}_i &= (11.700 - 0.049\text{age}_i - 0.343\text{gender}_i)\, I_{(1 \leq i \leq 78)} \\ &\quad + (11.129 - 0.042\text{age}_i - 0.461\text{gender}_i)\, I_{(79 \leq i \leq 184)}\end{aligned}$$

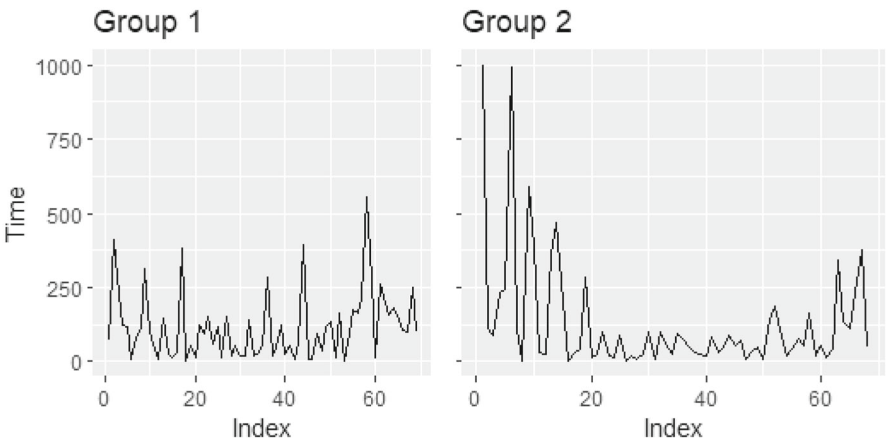

Fig. 4 Failure times

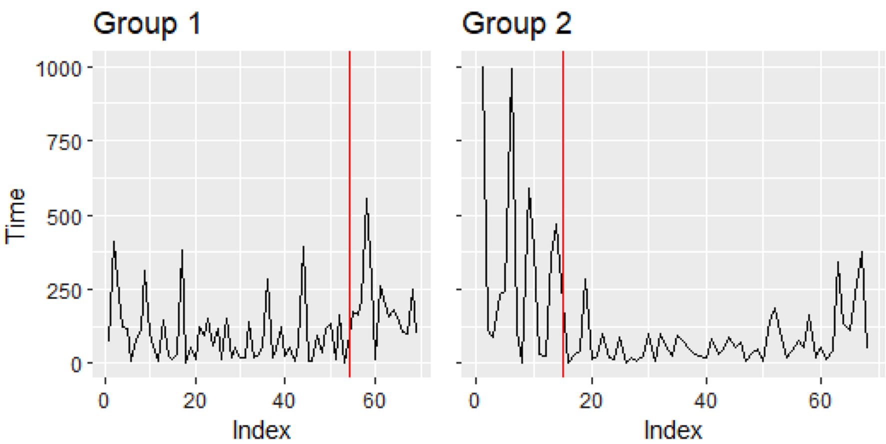

Fig. 5 Failure times data with change-point estimate

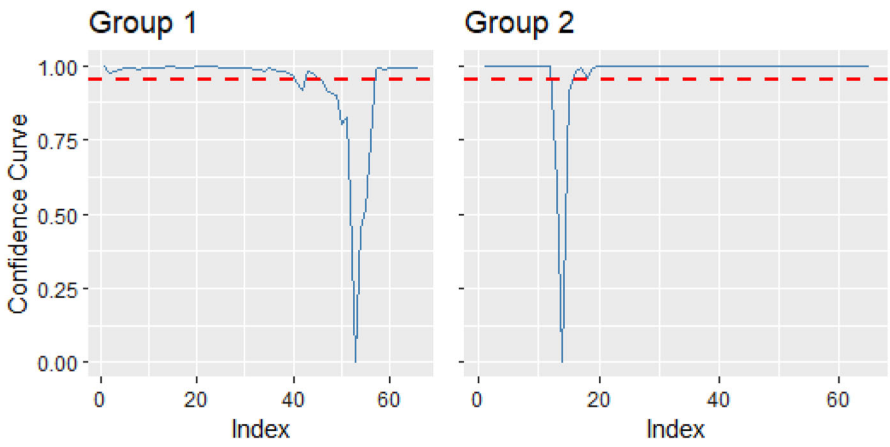

Fig. 6 95% confidence set of change point location

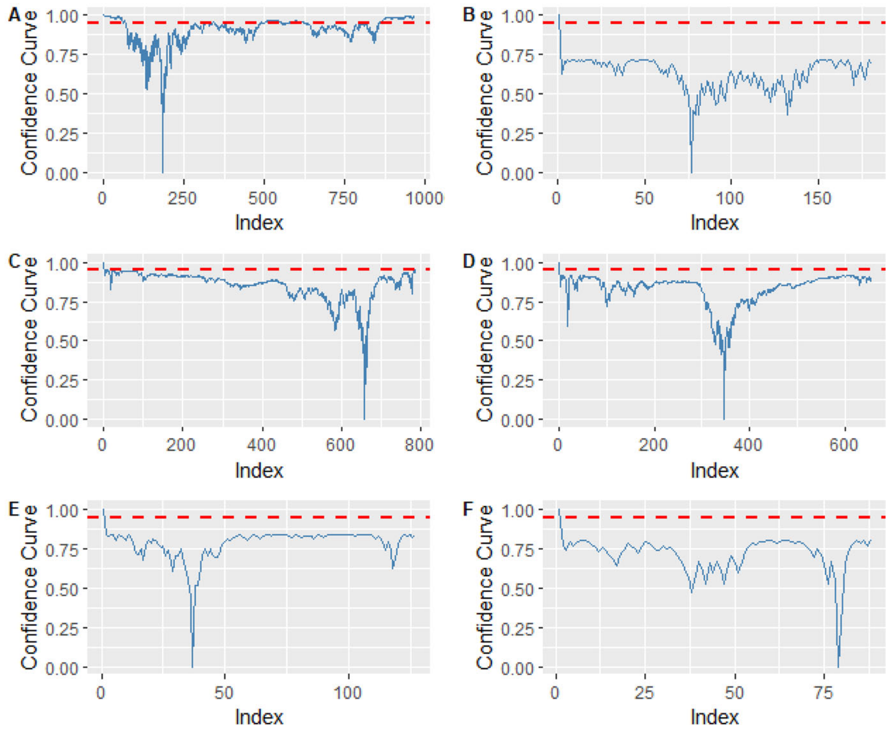

Fig. 7 **a** Confidence curve for change point estimate $\hat{k} = 184$, **b** Confidence curve for the subset ($2 \leq i \leq 183$), the change point estimate $\hat{k} = 78$, **c** Confidence curve for the subset ($185 \leq i \leq 972$), the change point estimate $\hat{k} = 842$, **d** Confidence curve for the subset ($185 \leq i \leq 841$), the change point estimate $\hat{k} = 532$, **e** Confidence curve for the subset ($843 \leq i \leq 972$), the change point estimate $\hat{k} = 881$, and **f** Confidence curve for the subset ($882 \leq i \leq 972$), the change point estimate $\hat{k} = 962$

$$
\begin{aligned}
&+ (11.697 - 0.062\text{age}_i + 0.054\text{gender}_i)\, I_{(185 \le i \le 532)} \\
&+ (10.241 - 0.029\text{age}_i - 0.120\text{gender}_i)\, I_{(533 \le i \le 842)} \\
&+ (10.738 - 0.045\text{age}_i - 0.099\text{gender}_i)\, I_{(843 \le i \le 881)} \\
&+ (10.433 - 0.042\text{age}_i 0.215\text{gender}_i)\, I_{(882 \le i \le 962)} \\
&+ (6.473 + 0.010\text{age}_i + 0.517\text{gender}_i)\, I_{(963 \le i \le 972)}
\end{aligned}
$$

However, [31]'s method considers randomly censored linear models without any change point.

5 Conclusion

In this paper, we propose a change point detection procedure in Linear Failure Rate distribution using the likelihood ratio test method in combining with the confidence distribution to construct the confidence sets of change locations, instead of providing point estimates only. Moreover, we consider the scenario of random censorship. Our proposed framework can be generalized to any combination of covariate distributions. We establish the asymptotic properties of the test statistic. Simulations are carried out under various conditions with different censoring distributions to demonstrate the advantages of the proposed method. Two real data applications are provided to illustrate the advantage of the proposed method.

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