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# Some new inequalities for the beta function and certain ratios of beta functions

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ABSTRACT

In this paper, we present various new inequalities and bounds for the beta function and some other related special functions. A variety of different approaches are used to derive these results, including various results from probability theory, in particular. The new upper and lower bounds for the beta function compare favorably to bounds discussed in various other works. Moreover, the methods of this paper can be used to obtain inequalities for other special functions not discussed in this work.



## 1. Introduction

The beta function is defined as:

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, \ y > 0.$$

The beta function can be expressed as the identity

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(x)$ is the gamma function

$$\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du.$$

In this paper, we are interested in discussing new bounds and inequalities for the beta function. A number of different approaches are used to derive these bounds and inequalities. The beta function has many applications in such diverse areas as probability/statistics, physics (such as string theory), economics, graph theory, reliability theory and actuarial science. First, we present previously published bounds and inequalities for the beta function which are most relevant to this work.

**Theorem 1.1** (*Dragomir et al. [1]*)**.** *The beta function $B(x, y)$ satisfies*

$$0 \le \frac{1}{xy} - B(x, y) \le \frac{1}{4}. \tag{1}$$

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The lower bound of (1) is sharp, but the upper bound is not. Later, Alzer [2] improved on this upper bound and obtained Theorem 1.2. below, which provides the best possible constant upper bound.

**Theorem 1.2** (*Alzer [2], p. 16*)**.** *For $x \geq 1$ and $y \geq 1$,*

$$0 \leq \frac{1}{xy} - B(x, y) \leq \beta,$$

*where $\beta \approx 0.08731$ is sharp.*

**Theorem 1.3** (*Cerone [3], Corollary 1, p.79*)**.** *For $x > 1$ and $y > 1$,*

$$\max\left\{\frac{1}{xy^x}, \frac{1}{yx^y}\right\} \leq B(x, y) \leq \min\left\{\frac{1}{y}\left[1 - \left(1 - \frac{1}{x}\right)^y\right], \frac{1}{x}\left[1 - \left(1 - \frac{1}{y}\right)^x\right]\right\}.$$

**Theorem 1.4** (*Cerone [3], Theorem 7, p. 80*)**.** *For $x > 1$ and $y > 1$,*

$$0 \leq \frac{1}{xy} - B(x, y) \leq \frac{x-1}{x\sqrt{2x-1}} \cdot \frac{y-1}{y\sqrt{2y-1}} \leq 0.090169437\ldots \tag{2}$$

where the upper bound is attained at

$$x = y = \frac{3 + \sqrt{5}}{2} \approx 2.618033988\ldots$$

In Remark 3 of Cerone [3], it is stated that the upper bound in (2) is better than the upper bound $\beta$ of Alzer [2] for a good majority of $x$ and $y$ values.

**Theorem 1.5** (*Cerone [3], Theorem 6, p. 80*)**.** *For $x > 1$ and $y > 1$,*

$$0 \leq \frac{1}{xy} - B(x, y) \leq 2\min\left\{A(x), A(y)\right\},$$

*where*

$$A(x) = \frac{x-1}{x^{\left(1+x/(x-1)\right)}}.$$

**Theorem 1.6** (*Cerone [3], Corollary 2, p. 81*)**.** *For $x > 1$ and $y > 1$,*

$$0 \leq \frac{1}{xy} - B(x, y) \leq 2\min\left\{C(x) \cdot C(y), b_A\right\},$$

*where*

$$C(x) = \frac{x-1}{x\sqrt{2x-1}}, \quad \text{and}$$

$$b_A = \max_{x \geq 1}\left(\frac{1}{x^2} - \frac{\Gamma^2(x)}{\Gamma(2x)}\right) = 0.08731\ldots$$

According to Grenié & Molteni [4] we have

$$B(x, y) \geq \frac{x^{x-1}y^{y-1}}{(x+y)^{x+y-1}} \cdot F(x, y).$$

It is proven that $\log F(x, y)$ has a Taylor expansion in every point $(x, y)$ with $x, y > 0$ whose coefficients can be explicitly computed in terms of values for the Hurwitz zeta function and that the Taylor expansion produces lower or upper bounds when truncated at even or odd order respectively, see, for example p.1430 of Grenié & Molteni [4]. Below Theorem 1.7 can be obtained when $F(x, y)$ is bounded from below by 1, that is, $F(x, y) \geq F(0^+, 0^+) = 1$ for every $x, y > 0$.

**Theorem 1.7** (*Grenié & Molteni [4]*)**.** *For $x > 0$ and $y > 0$,*

$$B(x, y) \geq \frac{x^{x-1}y^{y-1}}{(x+y)^{x+y-1}}.$$

For other inequalities and bounds for the beta function and other special functions, see, Alzer [5], Fisher [6], Alzer [7], McD.Mercer [8], Karatsuba & Vuorinen [9], Mitrinović [10] and Mitrinović et al. [11].

## 2. Preliminary results

To derive the new inequalities and bounds for the beta function, we shall need the following. The first few results below are from probability theory.

**Lemma 2.1.** *Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space and let $A$ and $B$ be independent, and continuous random variables. Let $F_A$ and $F_B$ be cumulative distribution functions (cdf) of $A$ and $B$, respectively, with corresponding probability density functions (pdf) given by $f_A = F_A'$ and $f_B = F_B'$. Then $C := A + B$ has cdf $F_C$ given by*

$$F_C(c) = P(C \le c) = \int_0^c F_A(c-b)dF_B(b) = \int_0^c F_A(c-b)f_B(b)db.$$

*Equivalently, by symmetry:*

$$F_C(c) = P(C \le c) = \int_0^c F_B(c-a)dF_A(a) = \int_0^c F_B(c-a)f_A(a)da.$$

*We say that $F_C$ is the convolution of $F_A$ and $F_B$ (or vice-versa) and write $F_C = F_A * F_B = F_B * F_A$.*

**Lemma 2.2** (*Brook [12], p. 171–173*)**.** *Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space and let $T$ be a random variable. Suppose there is a real number, $M > 0$ with $P(0 \le T \le M) = 1$. Let $\mu = E(T)$ be the mean or expected value of $T$. Then for all real values of $s$, $M_T(s) = E(e^{sT})$, the moment-generating function of $T$, is bounded above as:*

$$M_T(s) \le 1 - \frac{\mu}{M} + \frac{\mu}{M}e^{sM}.$$

**Lemma 2.3** (*Chernoff [13], Chernoff's Inequality*)**.** *Let $T$ be a random variable defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$. Then for all real numbers $a > 0$,*

$$P(T \ge a) \le \inf_{s>0} e^{-sa} \cdot M_T(s),$$

*where $M_T(s)$ is the moment-generating function of $T$ and the 'inf' is taken over all $s > 0$ with $M_T(s) < \infty$. (This is always the interval $(-\infty, t_0)$ for some $t_0$ with $0 \le t_0 \le \infty$, if $P(T \ge 0) = 1$).*

**Lemma 2.4** (*Karlin & Studden [14]*)**.** *Let $T$ be a random variable defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ with mean $\mu = E(T)$ and variance $\sigma^2 = V(T) < \infty$.*

*(a) Then for all real $z \ge 0$, the one-sided Chebyshev's inequality holds:*

$$P(T \ge \mu + z \cdot \sigma) \le \frac{1}{1+z^2}.$$

*Equivalently, if $a$ is a real number with $a \ge \mu$, then*

$$P(T \ge a) \le \frac{1}{1 + \left(\frac{a-\mu}{\sigma}\right)^2}.$$

*(b) Markov's inequality:*

$$P(T \ge a) \le \frac{\mu}{a}, \quad \text{if } a \ge \mu.$$

**Lemma 2.5** (*From [15], Theorem 3.1*)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be in $C^2([a,b])$. Let $H(t)$ be a continuous non-decreasing function $[a,b]$ with $H(a) = 0$, $H(b) = 1$ and $0 < H(t) < 1$ on $(a,b)$. Let*

$$L(t) = \begin{cases} \dfrac{\int_t^b (x-t)dH(x)}{1-H(t)} & \text{if } a \le t < b, \\ 0 & \text{if } t = b. \end{cases}$$

*Let*

$$q_1(t) = \inf\{f''(x) : t \le x \le t + L(t)\},$$

$$q_2(t) = \sup\{f''(x) : t \le x \le t + L(t)\},$$

$$L_1 = \frac{1}{2}\int_a^b q_1(t)\big(L(t)\big)^2 dH(t),$$

*and*

$$L_2 = \frac{1}{2}\int_a^b q_2(t)\big(L(t)\big)^2 dH(t).$$

*Then*

$$L_1 \le \int_a^b f(t)dH(t) - f\left(\int_a^b t dH(t)\right) \le L_2.$$

## 3. New beta function inequalities

We are now in a position to prove some new inequalities and bounds for the beta function $B(x, y)$.

**Theorem 3.1.** *Let $x > 0, y > 0$ satisfy: $xy < 1$. (This includes the cases $x \le 1$ and $y \le 1$, but also includes other choices for $x$ and $y$.) Then:*

$$B(x, y) \ge \frac{(x + y)\cdot(1 - \sqrt{xy})^2}{xy(x + 1)(y + 1)}. \tag{3}$$

**Proof.** Let $A$ be a random variable with cdf $F_A$ and pdf $f_A$ given by

$$F_A = \begin{cases} 0 & \text{if } a \le 0, \\ a^x & \text{if } 0 < a < 1, \\ 1 & \text{if } x \ge 1. \end{cases}$$

and

$$f_A(a) = \begin{cases} xa^{x-1} & \text{if } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Let $B$ be a random variable with cdf $F_B$ and pdf $f_B$ given by

$$F_B = \begin{cases} 0 & \text{if } b \le 0, \\ b^y & \text{if } 0 < b < 1, \\ 1 & \text{if } y \ge 1. \end{cases}$$

and

$$f_B(b) = \begin{cases} yb^{y-1} & \text{if } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Let $C = A + B$. Applying Lemma 2.1, $C$ has cdf

$$F_C(c) = P(C \le c) = \int_0^c F_A(1 - t)f_B(t)dt.$$

Letting $c = 1$,

$$F_C(1) = \int_0^1 (1 - t)^x \cdot yt^{y-1}dt = yB(y, x + 1) = yB(x + 1, y).$$

Thus, we obtain

$$P(A + B \le 1) = y \cdot B(x + 1, y). \tag{6}$$

Also, by Lemma 2.3, with $T = C = A + B,\ a = 1$, we get

$$\begin{aligned} P(A + B \ge 1) &\le \inf_{s>0} e^{-s}E(e^{s(A+B)}) \\ &= \inf_{s>0} e^{-s} \cdot E(e^{sA}) \cdot E(e^{sB}). \end{aligned} \tag{7}$$

By Lemma 2.2,

$$E(e^{sA}) \le \left(\frac{1}{x+1} + \frac{x}{x+1}e^s\right), \quad \text{and}$$
$$E(e^{sB}) \le \left(\frac{1}{y+1} + \frac{y}{y+1}e^s\right).$$

From (7), upon multiplication of these two inequalities, we get

$$P(A+B \ge 1) \le \inf_{s>0}\left[e^{-s} \cdot \left(\frac{1}{x+1} + \frac{x}{x+1}e^s\right)\left(\frac{1}{y+1} + \frac{y}{y+1}e^s\right)\right] \tag{8}$$

Simple differentiation of the expression in brackets of (8) with respect to $s$ gives that the optimal choice of $s > 0$ minimizing the right hand side (RHS) of (8) is

$$s = -\frac{1}{2}\big(\ln(xy)\big) > 0,$$

since $xy < 1$, with corresponding minimized value of $P(A+B \ge 1)$ of

$$P(A+B \ge 1) \le \frac{\left(\frac{1}{x+1} + \frac{x}{x+1}\sqrt{\frac{1}{xy}}\right) \cdot \left(\frac{1}{y+1} + \frac{y}{y+1}\sqrt{\frac{1}{xy}}\right)}{\sqrt{\frac{1}{xy}}}$$
$$\stackrel{\text{def.}}{=} w(x,y).$$

From (6), we get $P(A+B \le 1) = y \cdot B(x+1,y) \ge 1 - w(x,y)$. Since

$$B(x+1,y) = \frac{x}{x+y}B(x,y),$$

we obtain

$$B(x,y) \ge \frac{x+y}{xy}\big(1 - w(x,y)\big)$$
$$= \frac{(x+y)\big(1-\sqrt{xy}\big)^2}{xy(x+1)(y+1)}$$

which is (3). This completes the proof. ■

**Theorem 3.2.** *Suppose* $x > 0, y > 0$ *with* $\frac{x}{x+1} + \frac{y}{y+1} < 1$. *Then*

$$B(x,y) \ge \frac{x+y}{x} \cdot \frac{z^2}{(z^2+1)y},$$

*where*

$$z = \frac{1 - \frac{x}{x+1} - \frac{y}{y+1}}{\sqrt{\frac{x}{(x+1)^2(x+2)} + \frac{y}{(y+1)^2(y+2)}}}.$$

**Proof.** Proceeding as in the proof of Theorem 3.1, using

$$E(A+B) = \frac{x}{x+1} + \frac{y}{y+1},$$

and

$$Var(A+B) = \frac{x}{(x+1)^2(x+2)} + \frac{y}{(y+1)^2(y+2)},$$

and applying Lemma 2.4, part (a), with $T = A+B$, we obtain

$$P(A+B \ge 1) \le \frac{1}{1+z^2}.$$

Again applying Lemmas 2.1–2.3 and replacing $w(x, y)$ in Theorem 3.1 proof by $\frac{1}{1+z^2}$ and proceeding as we did there, with

$$z = \frac{1 - \frac{x}{x+1} - \frac{y}{y+1}}{\sqrt{\frac{x}{(x+1)^2(x+2)} + \frac{y}{(y+1)^2(y+2)}}}.$$

we obtain

$$B(x, y) \geq \frac{x+y}{xy}\left(1 - \frac{1}{1+z^2}\right) = \frac{x+y}{x} \cdot \frac{z^2}{(z^2+1)y}.$$

This completes the proof. ■

**Theorem 3.3.** *Suppose $x > 0, y > 0$ with $xy < 1$. Then*

$$B(x, y) \geq \frac{x+y}{x}\left[1 - \frac{x}{x+1} - \frac{y}{y+1}\right]. \tag{9}$$

**Proof.** Apply Lemma 2.4, part (b) with $T = A + B$. Then $\mu = E(T) = \frac{x}{x+1} + \frac{y}{y+1} \leq 1$, $a = 1$. Proceeding as in the proofs of Theorems 3.1 and 3.2, we obtain (9). ■

**Theorem 3.4.** *Suppose $x > 0, y > 0$ with $xy < 1$. Then*

$$B(x, y) \geq \frac{x+y}{xy}\left[1 - w(x, y)\right]. \tag{10}$$

*where*

$$w(x, y) = e^{-s}.\left[1 + s\left(\frac{x}{x+1}\right) + (e^s - 1 - s)\left(\frac{x}{x+2}\right)\right] \cdot \left[1 + s\left(\frac{y}{y+1}\right) + (e^s - 1 - s)\left(\frac{y}{y+2}\right)\right],$$

*and*

$$s = \frac{1}{2}\ln\left(\frac{1}{xy}\right).$$

**Proof.** Proceed as in the proofs of Theorems 3.1 and 3.2, with some modifications. For $s > 0$ and $0 < t \leq 1$, let

$$g(s, t) = \frac{e^{st} - 1}{st}.$$

For $s > 0$ and $t = 0$, let $g(s, t) = 1$. For each fixed $s$, $g(s, t)$ is a convex function of $t$. Consider the line segment joining $(0, g(s, 0))$ and $(1, g(s, 1))$. By convexity of $g$ in $t$ for all $s$, we obtain

$$\frac{e^{st} - 1}{st} \leq 1 + \frac{(e^s - 1 - s)t}{s},$$

which gives

$$e^{st} \leq 1 + st + (e^s - 1 - s)t^2.$$

Replacing $t$ by the values of the random variables $A$ and $B$ given earlier in the proof of Theorem 3.1, we obtain

$$E(e^{sA}) \leq 1 + s\left(\frac{x}{x+1}\right) + (e^s - 1 - s)\left(\frac{x}{x+2}\right), \quad \text{and}$$

$$E(e^{sB}) \leq 1 + s\left(\frac{y}{y+1}\right) + (e^s - 1 - s)\left(\frac{y}{y+2}\right).$$

Thus,

$$P(A + B \geq 1) \leq \frac{\left[1 + s\left(\frac{x}{x+1}\right) + (e^s - 1 - s)\left(\frac{x}{x+2}\right)\right] \cdot \left[1 + s\left(\frac{y}{y+1}\right) + (e^s - 1 - s)\left(\frac{y}{y+2}\right)\right]}{e^s}. \tag{11}$$

The value of $s$ minimizing the RHS of (11) satisfies

$$\frac{x + \theta x}{x + 1 + \theta x} + \frac{y + \theta y}{y + 1 + \theta y} = 1, \tag{12}$$

where $\theta = e^s - 1$. Differentiation of (12) with respect to $\theta$ gives $\theta = \sqrt{\frac{1}{xy}} - 1$, which gives $s = \frac{1}{2}\ln\left(\frac{1}{xy}\right) > 0$. Inserting this value of $s$ into (11) and using

$$B(x,y) \geq \frac{x+y}{xy} \cdot P(A+B \leq 1).$$

produces the lower bound given in (10). This completes the proof. ■

Theorems 3.1 and 3.2 presented lower bounds for $B(x,y)$. Each of the proofs of these theorems can be modified to give upper bounds also. Next, we present one such example.

**Theorem 3.5.** *Suppose $x > 0, y > 0$ with $\frac{1}{x+1} + \frac{1}{y+1} \leq 1$. Then*

$$B(x,y) \leq \frac{x+y}{xy} \cdot P(x,y),$$

*where*

$$P(x,y) = \frac{1}{1+\left(k(x,y)\right)^2}, \quad \text{and}$$

$$k(x,y) = \frac{1-\left(\frac{1}{x+1}+\frac{1}{y+1}\right)}{\sqrt{\frac{x}{(x+1)^2(x+2)}+\frac{y}{(y+1)^2(y+2)}}}.$$

**Proof.** Proceed as in the proofs of Theorems 3.1–3.3 with some modifications. Let $A$ and $B$ be random variables with distributions defined by (4) and (5) in the proof of Theorem 3.1. By considering the random variables $C = 1 - A$ and $D = 1 - B$, we get

$$P(A+B \geq 1) = 1 - P(C+D \geq 1).$$

Applying Lemma 2.4 to $T = C + D$ instead, we obtain, using $E(C) = \frac{1}{x+1}$ and $E(D) = \frac{1}{y+1}$;

$$P(C+D \geq 1) \leq \frac{1}{1+\left(k(x,y)\right)^2} = P(x,y). \tag{13}$$

Thus,

$$P(A+B \geq 1) = 1 - \frac{xy}{x+y}B(x,y). \tag{14}$$

From (13) and (14), we obtain

$$\frac{xy}{x+y}B(x,y) \leq P(x,y) = \frac{1}{1+\left(k(x,y)\right)^2}.$$

Solving for $B(x,y)$, we obtain the desired result. ■

Note that Theorem above can be used for $x \geq 1$ and $y \geq 1$, in particular, but can be used also for some choices of $x < 1$ or $y < 1$. Next we use the convolution idea in a different way.

**Theorem 3.6.** *Suppose $x > 0$ and $y > \frac{x+1}{x} > 1$. Then*

$$B(x,y) \leq \frac{1}{x\left(1+\left(k(x,y)\right)^2\right)},$$

*where*

$$k(x,y) = \frac{\frac{1}{y} - \frac{x}{x+1}}{\sqrt{\frac{x}{(x+1)^2(x+2)}+\frac{y}{(y+1)^2(y+2)}}}.$$

**Proof.** Let $A$ and $B$ be random variables with $A$ and $B$ having respective pdfs:

$$f_A(a) = \begin{cases} xa^{x-1} & \text{if } 0 < a < 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_B(b) = \begin{cases} (y-1)(1-b)^{y-2} & \text{if } 0 < b < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$B(x, y) = \frac{1}{x} \int_0^1 f_A(t) \cdot \big(1 - F_B(t)\big) dt, \tag{15}$$

where $F_B$ is the cdf of $B$. The integral in (15) is the convolution form of

$$P(B > A) = P(B - A > 0).$$

Thus

$$B(x, y) = \frac{1}{x} P(B - A > 0).$$

Applying Lemma 2.4 with $T = B - A$ and using $Z = k(x, y)$, we get the desired result. ■

**Theorem 3.7.** *Suppose $x > 0$ and $y > 0$, with $x \le y$.*

*(a) If $0 < x < 1$ and either $x + y \le 1$ or $x + y \ge 2$, then*

$$B(x, y) \ge (1 - x)^{x+y-1} \cdot \frac{\pi}{\sin(\pi x)}.$$

*(b) If $0 < x < 1$ and $1 < x + y < 2$, then*

$$B(x, y) \le (1 - x)^{x+y-1} \cdot \frac{\pi}{\sin(\pi x)}.$$

**Proof.** From the reflection formula of Euler, we have the integration formula

$$B(x, 1 - x) = \frac{\pi}{\sin(\pi x)}, \quad 0 < x < 1.$$

Then

$$\begin{aligned} B(x, y) &= \int_0^1 (1 - t)^{x+y-1} \cdot \left[ t^{x-1} (1 - t)^{-x} \right] dt \\ &= \left( \int_0^1 (1 - t)^{x+y-1} \cdot \left[ \frac{\sin(\pi x)}{\pi} t^{x-1} (1 - t)^{-x} \right] dt \right) \cdot \frac{\pi}{\sin(\pi x)}. \end{aligned} \tag{16}$$

The expression in brackets of the integral in (16) is a probability measure on $(0, 1)$. Also, $(1-t)^{x+y-1}$ is a concave function of $t$, if $1 < x + y < 2$ and is a convex function of $t$ otherwise. Applying Jensen's inequality to (16) and using the formula

$$\int_0^1 t^x (1 - t)^{-x} \cdot \frac{\sin(\pi x)}{\pi} dt = x,$$

we obtain both $(a)$ and $(b)$ of the theorem. ■

**Theorem 3.8.** *Suppose $x > 1$ and $y > 1$. Then*

$$B(x, y) \le \left( \frac{x - 1}{x + y - 2} \right)^{x-1} \left( \frac{y - 1}{x + y - 2} \right)^{y-1}. \tag{17}$$

**Proof.** Clearly,

$$\begin{aligned} B(x, y) &= \int_0^1 t^{x-1} (1 - t)^{y-1} dt \\ &\le \int_0^1 \sup_{0 \le t \le 1} t^{x-1} (1 - t)^{y-1} dt. \end{aligned} \tag{18}$$

The supremum is attained at $t = \left( \frac{x - 1}{x + y - 2} \right)$, by easy differentiation of the integral in (18) with respect to $t$. Then (17) is immediate. ■

**Theorem 3.9.** *Suppose $x_1 > 0, x_2 > 0$. Then*

*(a)*

$$\frac{B(x_2,x_2)}{B(x_1,x_1)} \le \left(\frac{1}{4}\right)^{x_2-x_1}, \quad \text{if } x_1 \le x_2$$

*(b)*

$$\frac{B(x_2,x_2)}{B(x_1,x_1)} \ge \left(\frac{1}{4}\right)^{x_2-x_1}, \quad \text{if } x_1 > x_2$$

**Proof.** We shall prove only $(a)$ since $(b)$ follows immediately from $(a)$. Let

$$g(w) = B(w,w) = \int_0^1 \big(t(1-t)\big)^{w-1} dw.$$

Differentiation of $g(w)$ gives

$$\begin{aligned} g'(w) &= \int_0^1 \big(t(1-t)\big)^{w-1}\big(\ln t + \ln(1-t)\big)dt \\ &= \int_0^1 \big(\ln t + \ln(1-t)\big)\left(\frac{1}{B(w,w)}\big(t(1-t)\big)^{w-1}\right)dt \cdot B(w,w). \end{aligned}$$

Since $\big(\ln t + \ln(1-t)\big)$ is a concave function of $t$, Jensen's inequality gives, using

$$\int_0^1 \frac{1}{B(w,w)} t^w (1-t)^{w-1} dt = \frac{1}{2},$$

we get

$$\frac{g'(w)}{g(w)} \le \ln\left(\frac{1}{4}\right). \tag{19}$$

Integration of (19) from $w = x_1$ to $w = x_2$ gives, upon exponentiation, the inequality in $(a)$. ■

**Remark 1.** A main use of Theorem 3.9 is to give both upper and lower bounds for $B(x,x)$ when $x$ is not an integer. When $x = m$ is a positive integer $m$,

$$B(m,m) = \frac{\Big[(m-1)!\Big]^2}{(2m-1)!}.$$

Letting $x_1 = x$ and $x_2 = \lceil x \rceil$ or $x_1 = \lfloor x \rfloor$ and $x_2 = x$ gives good bounds for $B(x,x)$ using $m = \lfloor x \rfloor$ or $m = \lceil x \rceil$. We can also take $x_1 = \frac{k}{2}$, or $x_2 = \frac{k}{2}$, where $k$ is an odd integer, since $B(k/2)$ can be written in terms of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. The bounds of Theorem 3.9 have an easier formulation when $x = y$ and are often easier to compute, not depending on possibly Hurwitz's zeta function. A similar statement applies at the end of Remark 3 below as well for the case $x \ne y$ in relation to Corollary 1 below.

**Theorem 3.10.** *Suppose $x \ge 3$ and $y > 0$. Then*

*(a)*

$$\begin{aligned} B(x,y) &\le \frac{1}{y}\left[\left(\frac{1}{1+y}\right)^{x-1} + \frac{\frac{1}{2}(x-1)\cdot\big((1+y)^{x-2}-1\big)}{(y+1)^{x-1}\cdot(y+2)}\right] \\ &\overset{\text{def.}}{=} u_1(x,y). \end{aligned} \tag{20}$$

*(b) If $x \ge 3$ and $y \ge 3$, then*

$$B(x,y) \le \min\Big\{u_1(x,y), u_1(y,x)\Big\}.$$

**Proof.** Since

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt,$$

we get

$$y \cdot B(x, y) = \int_0^1 t^{x-1}\Big[y(1-t)^{y-1}\Big]dt. \tag{21}$$

Applying Lemma 2.5 with $f(t) = t^{x-1}$ and

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1-(1-t)^y & \text{if } 0 < t < 1, \\ 1 & \text{if } t \geq 1, \end{cases} \tag{22}$$

and using the fact that the expression in brackets in (21) is a probability measure on (0, 1) with corresponding pdf $H'(t) = y(1-t)^{y-1}$, $0 \leq t \leq 1$, we find

$$L(t) = \frac{\int_0^1 (x-t)dH(t)}{1-H(t)} = \frac{1-t}{y+1}, \quad 0 \leq t \leq 1$$

and

$$\begin{aligned} q_2(t) &= (x-1)(x-2)(t+L(t))^{x-3} \\ &= (x-1)(x-2)\left(\frac{1+ty}{y+1}\right)^{x-3}. \end{aligned}$$

Thus,

$$L_2 = \frac{1}{2}\int_0^1 (x-1)(x-2)\left(\frac{1+ty}{y+1}\right)^{x-3}\left(\frac{1-t}{y+1}\right)^2 \cdot y(1-t)^{y-1}dt,$$

and

$$\begin{aligned} y \cdot B(x, y) &= \int_a^b f(t)dH(t) + L_2 \\ &= \left(\frac{1}{1+y}\right)^{x-1} + \frac{\frac{1}{2}(x-1)(x-2)}{(y+1)^{x-1}}\int_0^1 (1+yt)^{x-3} \cdot y(1-t)^{y+1}dt. \end{aligned} \tag{23}$$

Now $(1+yt)^{x-3}$ is increasing in $t$, since $x \geq 3$. Also, $y(1-t)^{y+1}$ is decreasing in $t$. Applying the Chebyshev–Grüss inequality to the integral in (23), we obtain

$$\begin{aligned} y \cdot B(x, y) &\leq \frac{1}{(1+y)^{x-1}} + \frac{\frac{1}{2}(x-1)(x-2)}{(y+1)^{x-1}}\left(\int_0^1 (1+yt)^{x-3}dt\right) \cdot \left(\int_0^1 y(1-t)^{y+1}dt\right) \\ &= \frac{1}{(1+y)^{x-1}} + \frac{\frac{1}{2}(x-1)}{(y+1)^{x-1}} \cdot \frac{1}{(y+2)} \cdot \big((1+y)^{x-2} - 1\big). \end{aligned}$$

which is the same as (20). This completes the proof of ($a$). Part ($b$) follows immediately from Part ($a$). ∎

**Remark 2.** Using the relation

$$B(x, y) = \left(\frac{(x+y+2)(x+y+1)(x+y)}{(x+2)(x+1)(x)}\right) \cdot B(x+3, y),$$

we can modify the bound in (20) to obtain an upper bound on $B(x, y)$ for $x > 0$ and $y > 0$, by first using the theorem to get an upper bound on $B(x+3, y)$.

**Theorem 3.11.** *For $x \geq 3$ and $y > 0$,*

*(a)*

$$\begin{aligned} B(x, y) &\geq \frac{1}{y}\left[\left(\frac{1}{1+y}\right)^{x-1} + \frac{\frac{1}{2}(x-1)y}{(y+1)^{x-1}(y+2)}\right] \\ &\overset{\text{def.}}{=} R_1(x, y). \end{aligned}$$

*(b) For $x \geq 3$ and $y \geq 3$,*

$$B(x, y) \geq \max\big\{R_1(x, y), R_1(y, x)\big\}.$$

**Proof.** The steps in the proof Theorem upper bound (20) are the same as the proof of Theorem above, except we use $L_1$ of Lemma 2.5 instead of $L_2$ and also we use $q_1(t) = (x-1)(x-2)t^{x-3}$ instead of $q_2(t)$. We omit details. ∎

Next, we present some inequalities for certain ratios of beta function values.

**Theorem 3.12.** *Suppose $y > 0$ and $0 \leq x_1 < x_2 < \infty$. Then*

$$\frac{B(x_2, y)}{B(x_1, y)} \leq x_2^{x_2}\left(\frac{1}{x_1}\right)^{x_1}\left(\frac{1}{x_2+y}\right)^{x_2+y}(x_1+y)^{x_1+y}. \tag{24}$$

**Proof.** Since $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$, taking a partial derivative, we get

$$\begin{aligned}\frac{\partial B}{\partial x} &= \int_0^1 (\ln t)\cdot t^{x-1}(1-t)^{y-1}dt \\ &= \left(\int_0^1\left[\frac{\Gamma(x+y)}{\Gamma(x)\Gamma(y)}t^{x-1}(1-t)^{y-1}\right]\cdot \ln t\ dt\right)\cdot B(x,y).\end{aligned} \tag{25}$$

Since $\ln t$ is a concave function of $t$ for $0 < t < 1$, Jensen's inequality applied to the integral in (25) gives

$$\frac{\partial B}{\partial x} \leq \left(\ln\left(\frac{x}{x+y}\right)\right)\cdot B(x,y).$$

Dividing by $B(x, y)$ and then integrating from $x = x_1$ to $x = x_2$ we obtain

$$\ln\left(\frac{B(x_2, y)}{B(x_1, y)}\right) \leq x_2 \ln x_2 - x_1 \ln x_1 - (x_2+y)\ln(x_2+y) + (x_1+y)\ln(x_1+y). \tag{26}$$

Exponentiation of both sides of (26) gives the upper bound in (24). ∎

**Corollary 1.** *Suppose $0 < x_1 \leq x_2 < \infty$ and $0 < y_1 \leq y_2 < \infty$. Let*

$$g(a,b,c) = b^b\left(\frac{1}{a}\right)^a\left(\frac{1}{b+c}\right)^{b+c}(a+c)^{a+c}, \quad \text{for } a > 0, b > 0, \text{ and } c > 0. \tag{27}$$

*Then:*

$$\frac{B(x_2, y_2)}{B(x_1, y_1)} \leq \left[g(x_1, x_2, y_2)\cdot g(y_1, y_2, x_1)\right]. \tag{28}$$

**Proof.** Apply Theorem 3.12. Then

$$\begin{aligned}B(x_2, y_2) &\leq g(x_1, y_1, y_2)B(x_1, y_2) \\ &\leq g(x_1, y_1, y_2)\left[g(y_1, y_2, x_1)\cdot B(x_1, y_1)\right],\end{aligned}$$

which is equivalent to (28). ∎

**Remark 3.** The real value of Corollary 1 is to obtain bounds for $B(x, y)$ when $x$ and $y$ are not positive integers. By choosing either $x_1 = \lfloor x \rfloor$ and $x_2 = x$ or $x_1 = x$ and $x_2 = \lceil x \rceil$ and similarly for $y_1$ and $y_2$, we can obtain both upper and lower bounds for $B(x, y)$ using

$$B(m,n) = \frac{(m-1)!(n-1)!}{(m+n-1)!},$$

if $m$ and $n$ are integers. In this way, excellent bounds for $B(x, y)$ are obtainable.

**Corollary 2.** *Suppose $x \geq 1, y \geq 1$ with $x \leq y$. Let*

$$u(x,y) = \frac{1}{y}x^x\left(\frac{1}{x+y}\right)^{x+y}(y+1)^{y+1}.$$

*Then*

$$B(x,y) \leq u(x,y). \tag{29}$$

**Proof.** The first half of (29) follows from $B(x, y) = B(y, x)$ and Theorem 3.12. The second half holds iff $u(x, y) \leq u(y, x)$, which is equivalent to

$$\left(\frac{x}{x+1}\right)^{x+1} < \left(\frac{y}{y+1}\right)^{y+1},$$

which follows from the fact that $(t+1)\left[\ln\left(\frac{t}{t+1}\right)\right]$ is increasing in $t,\ t \geq 0$. ∎

**Theorem 3.13.** *Let $0 < x_1 \le x_2 < \infty$. Then for $y > 0$,*

$$\frac{B(x_2, y)}{B(x_1, y)} \le \left(\frac{x_1 + y}{x_2 + y}\right)^y. \tag{30}$$

**Proof.** Since

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt.$$

Taking the partial derivative,

$$\frac{\partial B}{\partial x} = \int_0^1 t^{x-1}(1-t)^{y-1} \ln t \; dt.$$

Now $\ln t \le t - 1,\ 0 < t \le 1$, so

$$\begin{aligned}\frac{\partial B}{\partial x} &\le \int_0^1 t^{x-1}(1-t)^{y-1}(t-1) dt \\ &= \left(\frac{-y}{x+y}\right) \cdot B(x, y).\end{aligned}$$

which gives

$$\frac{\frac{\partial B}{\partial x}}{B(x, y)} = \left(\frac{-y}{x+y}\right). \tag{31}$$

Integrating (31) form $x = x_1$ to $x = x_2$ and then exponentiating gives upper bound (30). ∎

**Remark 4.** Earlier, in Theorem 3.12, we found upper bound (24):

$$\frac{B(x_2, y)}{B(x_1, y)} \le x_2^{x_2}\left(\frac{1}{x_1}\right)^{x_1}\left(\frac{1}{x_2+y}\right)^{x_2+y}(x_1+y)^{x_1+y}.$$

It can be shown that this bound is always better than the upper bound in Theorem 3.13 above, using the fact that $\left(1+\frac{y}{x}\right)^x$ is increasing in $x > 0$ for each value of $y > 0$. We omit the details here. However, the upper bound in (30) is much quicker to compute.

**Theorem 3.14.** *Let $x > 0$ and $y > 0$. then*

$$B(x, y) \ge \left(\frac{x}{x+y}\right)^x \cdot \left(\frac{e^{-x}}{\sqrt{x}}\right)\sqrt{2\pi}.$$

**Proof.** From 6.1.50 of Abramowitz & Stegun [16], p. 258, we have for $x > 0$:

$$\ln \Gamma(x) \ge \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi),$$

which gives

$$\Gamma(x) \ge x^{x-1/2} e^{-x} \sqrt{2\pi}. \tag{32}$$

Also, from 6.2.1 of Abramowitz & Stegun [16], p. 258, we have

$$B(x, y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} dt.$$

Since $e^{-t} \le \frac{1}{1+t}$, $t \ge 0$, we obtain

$$\begin{aligned}B(x, y) &\ge \int_0^\infty t^{x-1} e^{-t(x+y)} dt \\ &= \left(\frac{1}{x+y}\right)^x \Gamma(x).\end{aligned} \tag{33}$$

From (32) and (33) we get the desired result. ∎

**Theorem 3.15.** *Suppose* $0 < x \le y < \infty$. *Then*

$$B(x,y) \ge \frac{x^x y^y}{(x+y)^{x+y}} \cdot \sqrt{\frac{2\pi}{x}}. \tag{34}$$

**Proof.** From 6.1.50 of Abramowitz & Stegun [16], p. 258, we have for $x > 0$:

$$\ln \Gamma(x) \ge \left(x - \frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi),$$

which gives

$$\Gamma(x) \ge x^{x-1/2} e^{-x}\sqrt{2\pi}. \tag{35}$$

Also, from 6.2.1 of Abramowitz & Stegun [16], p. 258, we have

$$B(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}dt. \tag{36}$$

Since $\ln(1+t)$ is a concave function of $t \ge 0$, for $c \ge 0$,

$$\ln(1+t) \le \ln(1+c) + \left(\frac{t-c}{1+c}\right).$$

which gives

$$\frac{1}{1+t} \ge \frac{1}{1+c} e^{\left(\frac{c-t}{1+c}\right)}.$$

Thus,

$$\frac{1}{(1+t)^{x+y}} \ge \frac{1}{(1+c)^{x+y}} e^{\left(\frac{c-t}{1+c}\right)(x+y)}. \tag{37}$$

Inserting the RHS of (37) into the integral in (36) and integrating, we get

$$B(x,y) \ge \frac{1}{(1+c)^{x+y}} e^{\left(\frac{x+y}{1+c}\right)c} \cdot \left(\frac{1+c}{x+y}\right)^x \cdot \Gamma(x). \tag{38}$$

Applying (35), we get

$$B(x,y) \ge \left[\frac{1}{(1+c)^{x+y}} e^{\left(\frac{x+y}{1+c}\right)c} \cdot e^{-\left(\frac{x+y}{1+c}\right)t}\right] \cdot \left(x^{x-1/2} e^{-x}\sqrt{2\pi}\right). \tag{39}$$

Simple differentiation of the RHS of (38) with respect to $c$ maximizes the lower bound in (38). We get $c = \frac{x}{y}$ as the best choice. Substitution of this value of $c$ into (39) and simplifying the resulting expression gives the lower bound in (34). ■

**Theorem 3.16.** *For* $0 < x_1 < x_2 < \infty$,

$$\frac{B(x_2,y)}{B(x_1,y)} \ge \frac{x_1(x_2+y)}{x_2(x_1+y)} \cdot x_2^{x_2}\left(\frac{1}{x_1}\right)^{x_1}\left(\frac{1}{x_2+y}\right)^{x_2+y}(x_1+y)^{x_1+y}. \tag{40}$$

**Proof.** Since

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Taking the partial derivative,

$$\frac{\partial B}{\partial x} = \int_0^1 t^{x-1}(1-t)^{y-1}\ln t \; dt.$$

If $x > 1$, then

$$\frac{\partial B}{\partial x} = \int_0^1 t^{x-2}(1-t)^{y-1}(t\ln t)\,dt$$
$$= \left\{\int_0^1 \left[\frac{1}{B(x-1,y)}t^{x-2}(1-t)^{y-1}\right](t\ln t)\,dt\right\}\cdot B(x-1,y).$$

Since $(t\ln t)$ is a convex function of $t$, Jensen's inequality and the integration formula

$$\int_0^1 \frac{1}{B(x-1,y)}t^{x-1}(1-t)^{y-1}dt = \frac{x-1}{x+y-1},$$

gives:

$$\frac{\frac{\partial B}{\partial x}}{B(x,y)} \geq \ln\left(\frac{x-1}{x+y-1}\right). \tag{41}$$

Integration of (41) from $x = x_1$ to $x = x_2$ and simplifying, we get

$$\ln\left(\frac{B(x_2,y)}{B(x_1,y)}\right) \geq (x_2-1)\ln(x_2-1) - (x_1-1)\ln(x_1-1) - (x_2+y-1)\ln(x_2+y-1) + (x_1+y-1)\ln(x_1+y-1). \tag{42}$$

Exponentiation of (42) gives, for $x_2 > x_1 > 1$:

$$\frac{B(x_2,y)}{B(x_1,y)} \geq (x_2-1)^{x_2-1}\left(\frac{1}{x_1-1}\right)^{x_1-1}\left(\frac{1}{x_2+y-1}\right)^{x_2+y-1}(x_1+y-1)^{x_1+y-1}.$$

Using $B(x+1,y) = \dfrac{x}{x+y}B(x,y)$, for $x_2 > x_1 > 0$, this is equivalent to

$$\frac{B(x_2+1,y)}{B(x_1+1,y)} \geq x_2^{x_2}\left(\frac{1}{x_1}\right)^{x_1}\left(\frac{1}{x_2+y}\right)^{x_2+y}(x_1+y)^{x_1+y},$$

which gives,

$$\frac{B(x_2+1,y)}{B(x_1+1,y)} \geq \left(\frac{x_1}{x_1+y}\right)\left(\frac{x_2+y}{x_2}\right)x_2^{x_2}\left(\frac{1}{x_1}\right)^{x_1}\left(\frac{1}{x_2+y}\right)^{x_2+y}(x_1+y)^{x_1+y},$$

which is bound (40). This completes the proof. ■

Next, we give another upper bound for $\dfrac{B(x_2,y)}{B(x_1,y)}$.

**Theorem 3.17.** *Suppose* $0 < x_1 < x_2 < \infty$. *Let*

$$c = 1 + \frac{y\ln\left(\frac{x_2+y}{x_1+y}\right)}{x_1-x_2}, \quad a_1 = \ln c - 1, \quad \text{and } a_2 = \frac{1}{c}.$$

*Then*

$$\frac{B(x_2,y)}{B(x_1,y)} \leq e^{(a_1+a_2)(x_2-x_1)}\cdot\left(\frac{x_1+y}{x_2+y}\right)^{a_2 y}. \tag{43}$$

**Proof.** Since

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

Taking the partial derivative,

$$\frac{\partial B}{\partial x} = \int_0^1 t^{x-1}(1-t)^{y-1}\ln t\;dt.$$

Since $\ln t$ is a concave function of $t$, for $0 < t < 1$, $c > 0$,

$$\ln t \leq (\ln c - 1) + \frac{t}{c},$$

**Table 1**
Comparison of lower (LB) and upper (UB) bounds.

| | Theorems | $B(0.01, 0.01) = 199.9676$ | | $B(0.05, 0.2) = 24.6535$ | | $B(0.1, 0.8) = 10.3646$ | | $B(0.25, 3) = 2.8444$ | | $B(1.5, 1.5) = 0.3927$ | | $B(4, 8) = 0.0008$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | LB | UB | LB | UB | LB | UB | LB | UB | LB | UB | LB | UB |
| Existing bounds | 1.1 | – | – | – | – | – | – | – | – | 0.1944 | 0.4444 | – | 0.0313 |
| | 1.2 | – | – | – | – | – | – | – | – | 0.3571 | 0.4444 | – | 0.0313 |
| | 1.3 | – | – | – | – | – | – | – | – | 0.3629 | 0.5384 | 0.0001 | 0.1035 |
| | 1.4 | – | – | – | – | – | – | – | – | 0.3889 | 0.4444 | – | 0.0313 |
| | 1.5 | – | – | – | – | – | – | – | – | 0.2469 | 0.4444 | – | 0.0313 |
| | 1.6 | – | – | – | – | – | – | – | – | 0.3333 | 0.4444 | – | 0.0313 |
| | 1.7 | – | – | – | – | – | – | – | – | 0.1667 | – | 0.0002 | – |
| | [4], 1st/2nd order | 199.9675 | 199.9705 | 24.6415 | 24.7468 | 10.1063 | 10.8701 | – | – | 0.3899 | 0.4044 | – | – |
| Proposed bounds | 3.1 | 192.1576 | – | 16.0714 | – | 2.9222 | – | 0.0156 | – | – | – | – | – |
| | 3.2 | 197.9899 | – | 21.9665 | – | 7.0722 | – | 0.0975 | – | – | – | – | – |
| | 3.3 | 1.9604 | – | 3.9286 | – | 4.1818 | – | 0.6500 | – | – | – | – | – |
| | 3.4 | 195.4082 | – | 18.6585 | – | 4.0507 | – | 0.0259 | – | – | – | – | – |
| | 3.5 | – | – | – | – | – | – | – | – | – | 1.0323 | – | 0.0268 |
| | 3.6 | – | – | – | – | – | – | – | – | – | – | – | 0.0186 |
| | 3.7a | 101.0064 | – | 20.8701 | – | 10.2741 | – | 2.3257 | – | – | – | – | – |
| | 3.8 | – | – | – | – | – | – | – | – | – | 0.5000 | – | 0.0022 |
| | 3.10b | – | – | – | – | – | – | – | – | – | – | – | 0.0022 |
| | 3.11a | – | – | – | – | – | – | – | – | – | – | 0.0004 | – |
| | 3.11b | – | – | – | – | – | – | – | – | – | – | 0.0004 | – |
| | Cor - 2 | – | – | – | – | – | – | – | – | – | 0.4483 | – | 0.0014 |
| | 3.14 | 24.6454 | – | 9.8388 | – | 5.7575 | – | 2.0562 | – | 0.1615 | – | 0.0003 | – |
| | 3.15 | 24.7212 | – | 9.8918 | – | 5.7909 | – | 2.0766 | – | 0.2558 | – | 0.0006 | – |

which gives

$$\ln t \le a_1 + a_2 t. \tag{44}$$

Thus, from (44), we get

$$\frac{\frac{\partial B}{\partial x}}{B(x,y)} \le \left(a_1 + a_2\left(\frac{x}{x+y}\right)\right).$$

Integration from $x = x_1$ to $x = x_2$ and simplifying gives, for all $c > 0$:

$$\ln\left(\frac{B(x_2,y)}{B(x_1,y)}\right) \le (a_1 + a_2)(x_2 - x_1) - a_2 y \ln\left(\frac{x_2+y}{x_1+y}\right). \tag{45}$$

The value of $c > 0$ minimizing the RHS of (45) is found by easy differentiation to get

$$c = 1 + \frac{y\ln\left(\frac{x_2+y}{x_1+y}\right)}{x_1 - x_2},$$

which is a number in $(0, 1)$, since $\ln(1 + w) \le w$ for $w > 0$. Inserting this value of $c$ into (45) and then exponentiating and simplifying, we get upper bound (43). ■

**Remark 5.** If we proceed as in (27) we can get upper and lower bounds for $R = \frac{B(x_2,y_2)}{B(x_1,y_1)}$ as done earlier, using any of the upper or lower bounds for $R$ given earlier, when $x_1 < x_2,\ y_1 < y_2$.

## 4. Numerical study

In this section, we conduct a numerical study at various values of $x$ and $y$ values. Table 1 compares lower and upper bounds for the proposed theorems and some existing theorems. It can be clearly seen, in certain cases, our results provide excellent approximations for $B(x, y)$ and they are quite competitive with the bounds presented Dragomir et al. [1], Alzer [2], Cerone [3], and Grenié & Molteni [4].

## 5. Discussion

In this paper, we have proposed various new inequalities and bounds for the beta function and some other related special functions. It should be mentioned that the bounds for the beta function are reported in the literature (see, for example, Dragomir et al. [1], Alzer [2], and Cerone [3]) can be used when $x > 1$ and $y > 1$. The lower bounds presented in Theorems 3.1–3.4, and 3.7(a) are quite competitive with the lower bounds given in [4], 1st/2nd order. More specifically, our results have an easier formulation when $x = y$ and are often easier to compute, not depending on possibly Hurwitz's zeta function. As a result, our bounds are offered as a computationally simple alternative. In particular, the lower bound of Theorem 3.4 becomes quite good as $x$ and $y$ approach zero. Our upper bounds in Theorems 3.6, 3.8, and 3.10(b) are also quite competitive with the upper bounds of Dragomir et al. [1], Alzer [2], and Cerone [3]. When $x = 4$ and $y = 8$, for example, the lower bounds obtained in Theorems 3.11, 3.14, and 3.15 beat the lower bounds in Theorems 1.3 & 1.7. In most cases, our new lower and upper bounds are quite competitive with the existing bounds in the literature. In addition, we propose various inequalities for the ratio of beta functions. They are given in Theorems 3.9, 3.12, 3.13, 3.16, and 3.17. Our proposed new inequalities for the ratio of beta function can be used in linear preferential attachment models, a certain type of stochastic urn process. In particular, the fraction $P(k)$ of urns having $k$ balls in the limit of long time is $B(k+a,\gamma)/B(k_0+a,\gamma-1)$, for $k \geq k_0$. For more details, readers are referred to Medhi [17], Newman [18], and Dine [19]. Using our results, we can quickly find bounds on the $P(k)$. We have also developed an R package, **IneqBetaFun**, which can be found at https://github.com/suthakaranr/IneqBetaFun in which our theorems are implemented to allow readers to calculate bounds.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix

The proposed bounds are implemented as an R package called **IneqBetaFun**, freely available on GitHub. For instance, the bounds based on Theorems 3.1–3.17 and Corollaries 1 & 2 can be obtained as follows.

```
rm(list = ls())
library(devtools) # Make sure that the devtools library is loaded
install_github("suthakaranr/IneqBetaFun")

library(IneqBetaFun)
them_3_1(1/2, 1/2) # Theorem 3.1
them_3_2(1/2,1/2) # Theorem 3.2
them_3_3(1/2,1/2) # Theorem 3.3
them_3_4(1/2,1/2) # Theorem 3.4
them_3_5(2,2) # Theorem 3.5
them_3_6(2,2) # Theorem 3.6
them_3_7(0.25,0.5) #  Theorem 3.7: Case 1
them_3_7(0.5,1) #  Theorem 3.7: Case 2
them_3_8(2,2) # Theorem 3.8
them_3_10(4,2) # Theorem 3.10: Case 1
them_3_10(4,3.5) # Theorem 3.10: Case 2
them_3_11a(4,2) # Theorem 3.11: Case 1
them_3_11b(5,5) # Theorem 3.11: Case 2
corollary_2(1, 1.5) # Corollary 2
them_3_14(1,2) # Theorem 3.14
them_3_15(1,2) # Theorem 3.15

# Ratios of Beta Functions

them_3_9(0.5,0.75) # Theorem 3.9: Case 1: x1 <= x2
them_3_9(1.5,1)   # Theorem 3.9: Case 2: x1 > x2
them_3_17(1, 1.5, 2) # Theorem 3.17
corollary_ratio_1(1, 1.5,2, 3) # Corollary 1
them_3_13(1, 1.5,2) # Theorem 3.13
them_3_16(1, 1.5,2) # Theorem 3.16
them_3_17(1, 1.5,2) # Theorem 3.17
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