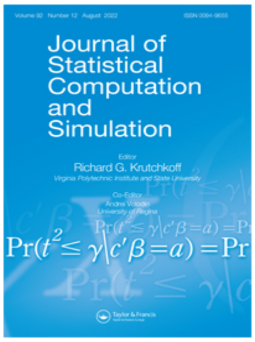

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Empirical-likelihood-based confidence intervals for quantile regression models with longitudinal data

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ABSTRACT

In this paper, we present three empirical likelihood (EL)-based inference procedures to construct confidence intervals for quantile regression models with longitudinal data. The traditional EL-based method suffers from an under-coverage problem, especially in small sample sizes. The proposed modified EL-based non-parametric methods including adjusted empirical likelihood (AEL), the transformed empirical likelihood (TEL), and the transformed adjusted empirical likelihood (TAEL) exhibit good finite sample performance over other existing procedures. Simulations are conducted to compare the performances of the proposed methods with the other methods in terms of coverage probabilities and average lengths of confidence intervals under different scenarios.

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Empirical likelihood; Confidence intervals; Quantile regression; Coverage probability

1. Introduction

In recent years, quantile regression (QR) has been widely used in many areas due to its attractive properties as opposed to the conventional ordinary least square (OLS) regression model. In their seminal work, Koenker and Bassett [1] introduced the QR approach as an alternative to the least square regression. In comparison to OLS, QR design to model the changes in the conditional quantiles of the response variable in relation to changes in the covariates. Several studies investigated the use of QR for the analysis of longitudinal data. Geraci and Bottai [2] proposed a linear model for QR that includes random effects to allow for the dependence between serial observations on the same subject. To estimate quantile functions with subject-specific fixed effects, Koenker [3] proposed the penalized interpretation of the classical random-effects estimator.

Empirical likelihood (EL) introduced by Owen [4] is a powerful nonparametric method. There is significant literature on the theoretical and practical application of the EL method. Further, the EL method holds appealing properties including range respecting, transformation-preserving, asymmetric confidence interval, Bartlett correctability, and better coverage probability for small samples, see, for example, [5,6]. Moreover, under mild

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regularity conditions, Owen [7,8] showed that the empirical likelihood ratio (ELR) statistic obeys the chi-square distribution asymptotically. This can be seen as the nonparametric extension of the well known Wilks' theorem. Because of this appealing property, the EL has been widely used for constructing confidence regions. There have been numerous studies that investigated the EL confidence intervals for quantile regression. Chen and Hall [9] derived the EL confidence intervals for the population quantiles without covariates. Whang [10] considered the smoothed EL (SEL) for quantile regression models with cross-sectional data. The EL for censored survival data proposed by Qin and Tsao [11]. Zhao and Chen [12] investigated the EL for the censored median regression model via nonparametric kernel estimation. Tang and Leng [13] considered the EL for QR in longitudinal data analysis. Wang and Zhu [14] developed two novel EL-based inference procedures for longitudinal data under the framework of quantile regression. Wang and Zhu [14] pointed out that the SEL procedure achieves higher-order accuracy by replacing the quantile score function with a smoothed counterpart. They used the blocking technique in order to accommodate the intra-subject correlation.

Chen et al. [15] pointed out the computation of the profile empirical likelihood function involves constrained maximization which requires that the convex hull of the estimating equation must have the zero vector as an interior point. This sometimes violates EL computation. As a result, the EL method suffers from an under-coverage problem. In order to rectify the problem, Chen et al. [15] proposed the adjusted empirical likelihood method (AEL) which ensures the existence of the solution in maximization problem and preserves the asymptotic optimality properties. Jing et al. [16] proposed a simple solution to the under-coverage problem especially for small sample sizes via transformed empirical likelihood (TEL). The transformed adjusted empirical likelihood (TAEL) proposed by Stewart and Ning [17] which combines the advantages of AEL and TEL.

In this paper, we proposed three novel EL-based procedures to construct confidence intervals for quantile regression models with longitudinal data based on the AEL, TEL, and TAEL. The rest of the paper is organized as follows. In Section 2, we briefly describe the quantile regression model for longitudinal data. The proposed AEL, TEL, and TAEL methods for longitudinal data in quantile regression models, main theoretical results and the construction of confidence regions are given in Section 3. Simulations to investigate the finite-sample performance of the proposed procedures and comparison between the proposed methods and other existing methods in terms of powers and average lengths of confidence sets are conducted in Section 4. A real data application is given in Section 5. Some discussion is provided in Section 6.

2. Methodology

Throughout this paper, we adopt notations similar to those of [14]. Let $\tau \in (0,1)$ be the quantile level of interest. The quantile regression model for longitudinal data is given below.

$$y_{ij} = x_{ij}^{\top}\beta_0 + e_{ij}, \quad i = 1,\dots,n, \quad j = 1,\dots,n_i, \tag{1}$$

where y_{ij} is the jth measurement of the ith subject, x_{ij} is the observed $p-$dimensional design vector, β_0 is a $p-$vector of unknown parameters, e_{ij} is the random error satisfying $P(e_{ij} < 0 \mid x_{ij}) = \tau$ for any i and j. The random errors are correlated within the same subject, but independent between subjects. Now, the quantile regression estimator $\hat{\beta}_\tau$ of β_0 is given as,

$$\min_{\beta \in \mathbb{B}} Q_n(\beta) = \arg\min_{\beta \in \mathbb{B}} \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{N} \rho_\tau(y_{ij} - x_{ij}^\top \beta), \tag{2}$$

where $\mathbb{B}$ is the parameter space and $\rho_\tau(u) = u\{\tau - I(u < 0)\}$ is the quantile loss function. For independent data, Koenker and Bassett [1] showed that $\hat{\beta}_\tau$ is $n^{1/2}-$ consistent and asymptotically normal. Under the above model assumptions, β_0 satisfies the following estimating equation:

$$E\big[x_{ij}\psi(y_{ij}, x_{ij}, \beta_0)\big] = 0, \tag{3}$$

where $\psi(y, x, \beta) = I(x^\top\beta - y > 0) - \tau$ is the quantile score function, and $I(\cdot)$ is the indicator function. You et al. [18] pointed out for longitudinal data the regular EL formulation cannot be used to derive the desired Wilk's theorem due to the correlation within subjects. The blocking technique proposed by Wang and Zhu [14] which treats $\psi(y_{ij}, x_{ij}, \beta_0)$, $j = 1, \dots, n_i$ as a whole unit in the development of EL. Let $X_i = (x_{i1}, \dots, x_{in_i})^\top$ be a $n_i \times p$ design matrix on the ith subject, $\psi_i(\beta) = (\psi(y_{i1}, x_{i1}, \beta), \dots, \psi(y_{in_i}, x_{in_i}, \beta))^\top$, and $Z_i(\beta) = X_i^\top \psi_i(\beta)$. Let $p_1 \dots, p_n$ be non-negative numbers satisfying $\sum_{i=1}^n p_i = 1$. The block empirical log-likelihood ratio for β is defined as

$$l(\beta) = \max\left\{ \sum_{i=1}^{n} \log(np_i) \,\middle|\, p_i \geq 0, \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i Z_i(\beta) = 0 \right\}. \tag{4}$$

The Lagrange multiplier method leads to

$$p_i(\beta) = \frac{1}{n}\left(\frac{1}{1 + \lambda(\beta)^\top Z_i(\beta)}\right), \tag{5}$$

where $\lambda(\beta)$ is $p-$dimensional Lagrange multiplier satisfying

$$\frac{1}{n}\sum_{i=1}^{n} \frac{Z_i(\beta)}{1 + \lambda(\beta)^\top Z_i(\beta)} = 0. \tag{6}$$

Thus, the empirical log-likelihood ratio statistic can be written as

$$l(\beta) = \sum_{i=1}^{n} \log\big(1 + \lambda(\beta)^\top Z_i(\beta)\big), \tag{7}$$

with $\lambda(\beta)$ satisfying (7). The equation (7) can be solved by the modified Newton-Raphson algorithm of Chen (2002). Thus, the maximum empirical likelihood estimator of β_0 as,

$$\hat{\beta}_{EL} = \arg\min_{\beta \in \mathbb{B}}\{-2l(\beta)\}. \tag{8}$$

Let h be a positive bandwidth parameter. Wang and Zhu [14] considered a smooth empirical likelihood (SEL) approach by approximating $\psi(\cdot)$ by a smooth function $\psi_h(\cdot)$ in order to achieve the higher-order accuracy. Define $G(x) = \int_{u<x} K(u)du$ and $G_h(x) = G(x/h)$,

where $K(\cdot)$ is bounded, compactly supported on $[-1, 1]$, and integrated to one. We approximate $\psi(\cdot)$ with $\psi_h(y_{ij}, x_{ij}, \beta) = G_h(x_{ij}^\top \beta - y_{ij}) - \tau$. Let $Z_{hi}(\beta) = X^\top \psi_h(\beta)$. The smooth empirical log-likelihood for β is defined as,

$$l_h(\beta) = \max \left\{ \sum_{i=1}^{n} \log(np_i) \middle| p_i \geq 0, \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i Z_{hi}(\beta) = 0 \right\}. \tag{9}$$

The maximum smooth empirical likelihood estimator of β_0 as,

$$\hat{\beta}_{SEL} = \arg\min_{\beta \in \mathbb{B}} \{-2l_h(\beta)\}. \tag{10}$$

Under some regularity conditions, Wang and Zhu [14] showed that $\hat{\beta}_{EL}$ and $\hat{\beta}_{SEL}$ have the same asymptotic distribution as $\hat{\beta}_Q$, as h goes to zero sufficiently fast when $n \longrightarrow \infty$.

3. Main results

In this section, we establish the theoretical properties of the EL quantile estimators $\hat{\beta}_{AEL}$, $\hat{\beta}_{TEL}$, and $\hat{\beta}_{TAEL}$. We assumes the following conditions used in [14,19] to establish the asymptotic properties of the proposed methods. As similar to [14], for simplicity, we consider a balanced design with $n_1 = \cdots = n_n = m$. First, we denote $F(u_1, \ldots, u_m \mid x)$ as the joint distribution function of $e_i = (e_{i1}, \ldots, e_{im})^\top$, and $F_j(u_j \mid x)$ as the marginal distribution function of e_{ij} conditional on $X_i = x$. Now, we define $f(u_1, \ldots, u_m \mid x)$ as the joint density of e_i, and $f_j(u_j \mid x)$ as the marginal density of e_{ij} with respect to the Legbesgue measure. Furthermore, let $\bar{f}(u \mid x) = \mathrm{diag}\{f_1(u_1 \mid x), \ldots, f_m(u_m \mid x)\}$, $S = E\{X_i^\top \bar{f}(0 \mid X_i) X_i\}$, and $\Sigma = E\{X_i^\top \psi_i(\beta_0)\psi_i(\beta_0)^\top X_i\}$, where $\psi_i(\beta_0) = (\psi(y_{i1}, x_{i1}, \beta_0), \ldots, \psi(y_{im}, x_{im}, \beta_0))^\top$ and let $r \geq 2$.

- (A1) Let $Y_i = (y_{i1}, \ldots, y_{im})^\top$. The $\{Y_i, X_i\}$, $i = 1, \ldots, n$ are i.i.d. random vectors.
- (A2) The parameter vector β_0 is an interior point of the parameter space $\mathbb{B}$, a compact subset of R^p.
- (A3) X_i has a bounded support, and matrices S and Σ are nonsingular.
  - (1) $f(u_1, \ldots, u_m | x)$ has a continuous partial derivative with respect to u_j, $j = 1, \ldots, m$.
  - (2) For all u_j in a neighborhood of 0 and almost every x, $f_j(u_j | x)$ exist, are bounded away from zero, and r times continuously differential with respect to u_j, $j = 1, \ldots, m$.

3.1. Modified empirical likelihood methods

3.1.1. Adjusted empirical likelihood

As discussed earlier, the EL method requires the convex hull of the estimating equation to contain a zero as an interior point. Owen [8] suggested assigning $-\infty$ to the empirical log-likelihood ratio statistic if the solution doesn't exist. Chen et al. [15] suggested adding a pseudo term to ensure that the zero-vector is within the convex hull. Let $Z_i = Z_i(\beta)$ for $i = 1, \ldots, n$. Reprising [15], for any given β and some positive constant a_n. Let $\bar{Z}_n =$

$\bar{Z}_n(\beta) = \frac{1}{n}\sum_{i=1}^{n} Z_i$ for any given β. We define an additional term,

$$Z_{n+1} = Z_{n+1}(\beta) = \frac{-a_n}{n}\sum_{i=1}^{n} Z_n = -a_n\bar{Z}_n. \tag{11}$$

Then $l(\beta)$ can be adjusted as

$$l^*(\beta) = \max\left\{\sum_{i=1}^{n+1}\log((n+1)p_i)\,\middle|\, p_i \geq 0, \sum_{i=1}^{n+1} p_i = 1, \sum_{i=1}^{n+1} p_i Z_i(\beta) = 0\right\}. \tag{12}$$

Chen et al. [15] showed that, as $n \longrightarrow \infty$, $-2l^*(\beta) \longrightarrow \chi^2_{1-\alpha}(p)$ in distribution with $a_n = o_p(n^{2/3})$, where p is the dimension of the vector (x_{ij}). First we will show that the AEL has the same asymptotic properties as the unadjusted EL.

Theorem 3.1: *Let β_0 be the true parameter that satisfies $E\{Z_i(\beta)\} = 0$ and the $\Sigma(\beta_0) = E\{X_i^\top \psi_i(\beta_0)\psi_i(\beta_0)^\top X_i\}, i = 1, \ldots, n$, is of full rank. Let $l^*(\beta)$ be the adjusted profile log-likelihood ratio function defined in* (12) *and $a_n = o_p(n^{2/3})$. As $n \longrightarrow \infty$, we have*

$$-2l^*(\beta_0) \longrightarrow \chi^2(p).$$

in distribution, where p is the dimension of the vector (x_{ij}).

Proof: Let $\lambda(\beta)$ be the solution to

$$\sum_{i=1}^{n+1}\frac{Z_i(\beta)}{1+\lambda^\top Z_i(\beta)} = 0 \tag{13}$$

We first show that $\lambda = O_p(n^{-1/2})$. Let $Z^* = \max_{1\leq\leq n}\|Z_i\| = o_p(n^{1/2})$. Let $\rho = \|\lambda\|$ and $\hat{\lambda} = \lambda/\rho$. Multiplying $\hat{\lambda}/n$ to both sides gives,

$$\begin{aligned}
0 &= \frac{\hat{\lambda}}{n}\sum_{i=1}^{n+1}\frac{Z_i(\beta)}{1+\hat{\lambda}^\top Z_i(\beta)} \\
&= \frac{\hat{\lambda}}{n}\sum_{i=1}^{n+1} Z_i - \rho\sum_{i=1}^{n+1}\frac{(\hat{\lambda}^\top Z_i)^2}{(1+\rho\hat{\lambda}^\top Z_i)} \\
&\leq \hat{\lambda}^\top \bar{Z}_n(1-a_n/n) - \frac{\rho}{n(1+\rho Z^*)}\sum_{i=1}^{n}(\hat{\lambda}^\top Z_i)^2 \\
&= \hat{\lambda}^\top \bar{Z}_n - \frac{\rho}{n(1+\rho Z^*)}\sum_{i=1}^{n}(\hat{\lambda}^\top Z_i)^2 + O_p(n^{-2/3}a_n).
\end{aligned} \tag{14}$$

Using the assumption on variance, we have

$$\hat{\Sigma}(\hat{\beta}) = \frac{1}{n}\sum_{i=1}^{n} Z_i(\hat{\beta})Z_i(\hat{\beta})^\top. \tag{15}$$

Using the Lemma 4.1 given in [20] gives,

$$\frac{1}{n}\sum_{i=1}^{n} Z_i(\beta) = \frac{1}{n}\sum_{i=1}^{n} Z_i(\beta_0) + \frac{1}{n}\sum_{i=1}^{n} X_i^\top \bar{f}(0|X_i)X_i(\beta-\beta_0) + o_p(n^{-\delta}). \tag{16}$$

By (16) and similar arguments of the proof of Lemma 1 in [21], we have,

$$\begin{aligned}\lambda(\beta) &= \left\{\frac{1}{n}\sum_{i=1}^{n} Z_i(\beta)Z_i(\beta)^\top\right\}^{-1} \frac{1}{n}\sum_{i=1}^{n} Z_i(\beta) + o_p(n^{-\delta}) \\ &= \Sigma^{-1} n^{-1}\sum_{i=1}^{n} Z_i(\beta) + O_p(n^{-\delta}).\end{aligned} \tag{17}$$

For any given $\epsilon > 0$,

$$\frac{1}{n}\sum_{i=1}^{n}\{\lambda(\beta)^\top Z_i(\beta)\}^2 \geq 1-\epsilon. \tag{18}$$

Therefore, as long as $a_n = o_p(n)$, we get (15), which implies that,

$$\frac{\rho}{(1+\rho Z^*)} \leq \hat{\lambda}^\top \frac{\bar{Z}_n}{(1-\epsilon)} = O_p(n^{-1/2}). \tag{19}$$

Thus, we get $\rho = O_p(n^{-1/2})$ and hence $\lambda = O_p(n^{-1/2})$. Now consider,

$$\begin{aligned}0 &= \frac{1}{n}\sum_{i=1}^{n+1} \frac{Z_i(\beta)}{1+\lambda^\top Z_i(\beta)} \\ &= \bar{Z}_n - \lambda^\top \hat{V}_n + o_p(n^{-1/2}).\end{aligned} \tag{20}$$

where $\hat{V}_n = \frac{1}{n}\sum_{i=1}^{n} Z_i(\beta)Z_i(\beta)^\top$. Hence, when $n \longrightarrow \infty, \lambda = \hat{V}_n^{-1}\bar{Z}_n + o_p(n^{-1/2})$. Now, we expand l^* as follows

$$\begin{aligned}-2l^*(\beta_0) &= 2\sum_{i=1}^{n+1}\log(1+\lambda^\top Z_i(\beta)) \\ &= 2\sum_{i=1}^{n+1}\left\{\lambda^\top Z_i(\beta) - \frac{\left(\lambda^\top Z_i(\beta)\right)^2}{2}\right\} + o_p(1).\end{aligned} \tag{21}$$

Substituting the expansion of λ, we get that

$$\begin{aligned}-2l^*(\beta_0) &= n\bar{Z}_n^\top \hat{V}_n^{-1}\bar{Z}_n + o_p(1) \\ &\xrightarrow{d} \chi^2(p).\end{aligned} \tag{22}$$

■

Table 1. Estimated coverage probabilities (CP) of confidence intervals for β_1, β_1, and the average lengths (AL) of confidence intervals from different methods in Model 1.

			β_0		β_1	
n	τ	Method	CP	AL	CP	AL
30	0.5	EL	0.9260	0.6282	0.9355	0.2095
		SEL	0.9340	0.6348	0.9410	0.2046
		AEL	0.9320	0.6395	0.9395	0.2149
		TEL	0.9395	0.6607	0.9510	0.2282
		TAEL	0.9425	0.6715	0.9560	0.2358
	0.7	EL	0.9180	0.6434	0.9390	0.2226
		SEL	0.9210	0.6482	0.9205	0.2131
		AEL	0.9230	0.6545	0.9445	0.2285
		TEL	0.9315	0.6771	0.9530	0.2423
		TAEL	0.9360	0.6885	0.9605	0.2502
50	0.5	EL	0.9270	0.4966	0.9465	0.1658
		SEL	0.9280	0.4988	0.9365	0.1603
		AEL	0.9295	0.5056	0.9540	0.1703
		TEL	0.9335	0.5116	0.9600	0.1737
		TAEL	0.9360	0.5201	0.9635	0.1788
	0.7	EL	0.9250	0.5124	0.9330	0.1742
		SEL	0.9380	0.5148	0.9320	0.1692
		AEL	0.9295	0.5216	0.9385	0.1789
		TEL	0.9320	0.5279	0.9415	0.1823
		TAEL	0.9380	0.5368	0.9480	0.1877
100	0.5	EL	0.9465	0.3568	0.9495	0.1171
		SEL	0.9395	0.3568	0.9345	0.1151
		AEL	0.9490	0.3635	0.9555	0.1203
		TEL	0.9485	0.3624	0.9550	0.1197
		TAEL	0.9505	0.3688	0.9595	0.1229
	0.7	EL	0.9410	0.3680	0.9435	0.1242
		SEL	0.9395	0.3677	0.9395	0.1215
		AEL	0.9450	0.3748	0.9525	0.1273
		TEL	0.9440	0.3735	0.9505	0.1267
		TAEL	0.9460	0.3801	0.9575	0.1300

3.1.2. Transformed empirical likelihood

Wang and Zhu [14] considered Bartlett correction for the smoothed empirical likelihood in order to improve coverage accuracy. However, Corcoran et al. [22] noted that for finite sample applications the Bartlett correction factor may be difficult to estimate. Jing et al. [16] proposed a procedure to overcome the under-coverage problem in EL which requires a simple transformation of the original EL. Indeed, Jing et al. [16] approach gives substantially more accurate confidence regions without adding theoretical or computational complexity. For a constant $\gamma \in [0, 1]$, we define

$$Z_t(l(\beta), \gamma) = l(\beta) \times \max\{1 - l(\beta)/n, 1 - \gamma\}, \tag{23}$$

and refer to $Z_t(l(\beta); \gamma)$ as the truncated quadratic transformation of $l(\beta)$ defined in (4). Following, Jing et al. [16], we set $\gamma = 1/2$. Thus, the transformed empirical log-likelihood ratio can be defined as follows.

$$l_t(\beta) = Z_t(l(\beta), \gamma = 1/2) = l(\beta) \times \max\{1 - l(\beta)/n, 1/2\}. \tag{24}$$

Table 2. Estimated coverage probabilities (CP) of confidence intervals for β_1, β_1, and the average lengths (AL) of confidence intervals from different methods in Model 2.

			β_0		β_1	
n	τ	Method	CP	AL	CP	AL
30	0.5	EL	0.9350	0.2138	0.9340	0.1887
		SEL	0.9370	0.2108	0.9335	0.1848
		AEL	0.9420	0.2193	0.9395	0.1934
		TEL	0.9490	0.2315	0.9565	0.2037
		TAEL	0.9555	0.2386	0.9610	0.2095
	0.7	EL	0.9305	0.2185	0.9300	0.1957
		SEL	0.9330	0.2154	0.9325	0.1910
		AEL	0.9360	0.2238	0.9345	0.1999
		TEL	0.9530	0.2363	0.9450	0.2106
		TAEL	0.9580	0.2431	0.9510	0.2161
50	0.5	EL	0.9345	0.1672	0.9480	0.1488
		SEL	0.9370	0.1652	0.9475	0.1461
		AEL	0.9410	0.1716	0.9515	0.1525
		TEL	0.9470	0.1749	0.9570	0.1552
		TAEL	0.9565	0.1800	0.9605	0.1593
	0.7	EL	0.9400	0.1721	0.9340	0.1546
		SEL	0.9390	0.1701	0.9370	0.1520
		AEL	0.9450	0.1767	0.9410	0.1583
		TEL	0.9500	0.1799	0.9460	0.1610
		TAEL	0.9565	0.1849	0.9515	0.1652
100	0.5	EL	0.9445	0.1181	0.9405	0.1059
		SEL	0.9485	0.1173	0.9395	0.1049
		AEL	0.9500	0.1211	0.9450	0.1087
		TEL	0.9480	0.1205	0.9440	0.1081
		TAEL	0.9565	0.1237	0.9485	0.1110
	0.7	EL	0.9495	0.1220	0.9410	0.1111
		SEL	0.9510	0.1211	0.9420	0.1097
		AEL	0.9560	0.1252	0.9470	0.1138
		TEL	0.9545	0.1246	0.9465	0.1132
		TAEL	0.9590	0.1280	0.9525	0.1162

The corresponding transformed empirical log-likelihood ratio, denoted by $l(\beta)$, is

$$l_t(\beta) = \begin{cases} l(\beta)[1 - l(\beta)/n] & \text{if } l(\beta) \leq n/2, \\ l(\beta)/2 & \text{if } l(\beta) > n/2. \end{cases} \tag{25}$$

Jing et al. [16] pointed out that the TEL shares the same asymptotic properties with the EL. For more details readers are encouraged to look into the original reference [16].

Theorem 3.2: *Let β_0 be the true parameter that satisfies $E\{Z_i(\beta)\} = 0$ and the $\Sigma(\beta_0) = E\{X_i^\top \psi_i(\beta_0)\psi_i(\beta_0)^\top X_i\}, i = 1, \ldots, n$, is of full rank. Let $l_t(\beta)$ be the adjusted profile log-likelihood ratio function defined in (24) and $a_n = o_p(n^{2/3})$. As $n \longrightarrow \infty$, we have*

$$-2l_t(\beta_0) \longrightarrow \chi^2(p).$$

in distribution.

Proof: We consider the same arguments used in [16]. We will look at four criteria separately.

Table 3. Estimated coverage probabilities (CP) of confidence intervals for β_1, β_1, and the average lengths (AL) of confidence intervals from different methods in Model 3.

			β_0		β_1	
n	τ	Method	CP	AL	CP	AL
30	0.5	EL	0.9330	0.3831	0.9330	0.2671
		SEL	0.9315	0.3756	0.9355	0.2614
		AEL	0.9375	0.3939	0.9410	0.2745
		TEL	0.9510	0.4185	0.9535	0.2920
		TAEL	0.9590	0.4327	0.9570	0.3015
	0.7	EL	0.9310	0.5495	0.9310	0.3722
		SEL	0.9330	0.5407	0.9340	0.3742
		AEL	0.9385	0.5640	0.9365	0.3822
		TEL	0.9480	0.5992	0.9490	0.4061
		TAEL	0.9525	0.6195	0.9520	0.4193
50	0.5	EL	0.9445	0.2985	0.9400	0.2060
		SEL	0.9455	0.2939	0.9395	0.2033
		AEL	0.9510	0.3064	0.9455	0.2116
		TEL	0.9565	0.3122	0.9485	0.2158
		TAEL	0.9585	0.3217	0.9525	0.2226
	0.7	EL	0.9450	0.4231	0.9380	0.2857
		SEL	0.9480	0.4182	0.9435	0.2824
		AEL	0.9510	0.4347	0.9455	0.2939
		TEL	0.9535	0.4432	0.9500	0.2997
		TAEL	0.9590	0.4565	0.9540	0.3085
100	0.5	EL	0.9440	0.2103	0.9460	0.1451
		SEL	0.9440	0.2084	0.9505	0.1436
		AEL	0.9500	0.2158	0.9530	0.1489
		TEL	0.9520	0.2148	0.9525	0.1482
		TAEL	0.9555	0.2207	0.9575	0.1522
	0.7	EL	0.9455	0.2958	0.9540	0.2014
		SEL	0.9495	0.2932	0.9530	0.1999
		AEL	0.9520	0.3034	0.9600	0.2067
		TEL	0.9590	0.3019	0.9590	0.2057
		TAEL	0.9610	0.3099	0.9640	0.2114

- (C_1) $0 \leq l_t(\beta) \leq l(\beta)$
- (C_2) $l_t(\beta)$ is a monotonically increasing function of $l(\beta)$
- (C_3) $l_t(\beta_0) = l(\beta_0) + o_p(1)$
- (C_4) For any $\tau_1 \in [0, +\infty)$ the level-τ_1 contour of $l_t(\beta)$, $\{\beta : l_t(\beta) = \tau_1\}$ is the same in shape as some level-τ_2 contour

We evaluate criteria (C1) through (C4) given below.

- (C_1) We can easily see that from the original empirical log-likelihood $l(\beta)(\geq 0)$. This implies that

$$0 < \max\{1 - l(\beta)/n, 1/2\} \leq 1. \tag{26}$$

  Hence, $0 \leq l_t(\beta) \leq l(\beta)$.
- (C_2) For $l(\beta) \in [0, n/2]$, we have $l_t(\beta) = l(\beta) \times \max\{1 - l(\beta)/n, 1/2\}$. Specifically, $l_t(\beta)$ is a strictly monotonically increasing function of $l(\beta)$ over the interval $[0, n/2]$. Thus for $l(\beta) > n/2$, we have $l_t(\beta) = l(\beta)/2$. This is also a strictly monotonically increasing function of $l(\beta)$. Therefore, $l_t(\beta)$ is non-negative, continuous, and strictly monotonically increasing over $l(\beta) \in [0, +\infty]$.

Table 4. Estimated coverage probabilities (CP) of confidence intervals for β_1, β_1, and the average lengths (AL) of confidence intervals from different methods in Model 4.

			β_0		β_1	
n	τ	Method	CP	AL	CP	AL
30	0.5	EL	0.9285	1.5710	0.9330	1.0047
		SEL	0.9320	1.5792	0.9320	0.9984
		AEL	0.9385	1.6125	0.9415	1.0328
		TEL	0.9500	1.7076	0.9545	1.0973
		TAEL	0.9560	1.7641	0.9600	1.1361
	0.7	EL	0.9250	2.0790	0.9315	1.3958
		SEL	0.9310	2.1280	0.9280	1.3828
		AEL	0.9375	2.1310	0.9355	1.4332
		TEL	0.9480	2.2549	0.9470	1.5206
		TAEL	0.9510	2.3233	0.9550	1.5704
50	0.5	EL	0.9270	1.2276	0.9425	0.7716
		SEL	0.9265	1.2273	0.9425	0.7665
		AEL	0.9335	1.2609	0.9510	0.7926
		TEL	0.9365	1.2862	0.9565	0.8079
		TAEL	0.9435	1.3240	0.9595	0.8327
	0.7	EL	0.9310	1.6868	0.9440	1.1086
		SEL	0.9355	1.6979	0.9435	1.1017
		AEL	0.9480	1.7310	0.9520	1.1380
		TEL	0.9540	1.7637	0.9555	1.1598
		TAEL	0.9655	1.8126	0.9675	1.1955
100	0.5	EL	0.9325	0.8628	0.9445	0.5382
		SEL	0.9325	0.8617	0.9455	0.5359
		AEL	0.9395	0.8851	0.9500	0.5524
		TEL	0.9380	0.8807	0.9485	0.5497
		TAEL	0.9445	0.9049	0.9565	0.5649
	0.7	EL	0.9410	1.2085	0.9500	0.7924
		SEL	0.9480	1.2078	0.9470	0.7902
		AEL	0.9535	1.2400	0.9560	0.8136
		TEL	0.9580	1.2337	0.9540	0.8095
		TAEL	0.9665	1.2669	0.9640	0.8312

- (C_3) Wang and Zhu [14] showed that the limiting distribution of $l(\beta_0)$ is $\chi^2(p)$, distribution, we have that $l(\beta_0) = O_p(1)$. Thus with probability tending to unity we have $l(\beta_0) \le n/2$. Thus, it follows that for all asymptotic discussions we may simply assume that $l_t(\beta_0) = l(\beta_0) \times \max\{1 - l(\beta_0)/n, 1/2\}$. Using this fact and that $l(\beta_0) = O_p(1)$ gives us (C_3).
- (C_4) For a level-τ_1 contour of the transformed empirical log-likelihood ratio $\{\beta : l_t(\beta) = \tau_1\}$, as $l_t(\beta)$ is a strictly monotonically increasing function of $l(\beta)$, let $\tau_2 = l_t^{-1}(\tau_1)$, then $\{\beta : l_t(\beta) = \tau_1\} = \{\beta : l(\beta) = \tau_2\}$. Further, as $l(\beta)$ typically has a unique minimum at $\tilde{\beta}$, the second part of (C4) also follows from the monotonicity of $l_t(\beta)$.

■

3.1.3. Transformed adjusted empirical likelihood

Transformed adjusted empirical likelihood (TAEL) is a combination of AEL and TEL methods proposed by Stewart and Ning [17]. The TAEL method comprises the advantages of AEL and TEL. Let $Z_i = Z_i(X_i, \beta)$ for $i = 1, \ldots, n$. For a constant $\gamma \in [0, 1]$, we define

$$Z_t^*(l^*(\beta), \gamma) = l^*(\beta) \times \max\{1 - l^*(\beta)/n, 1 - \gamma\}. \tag{27}$$

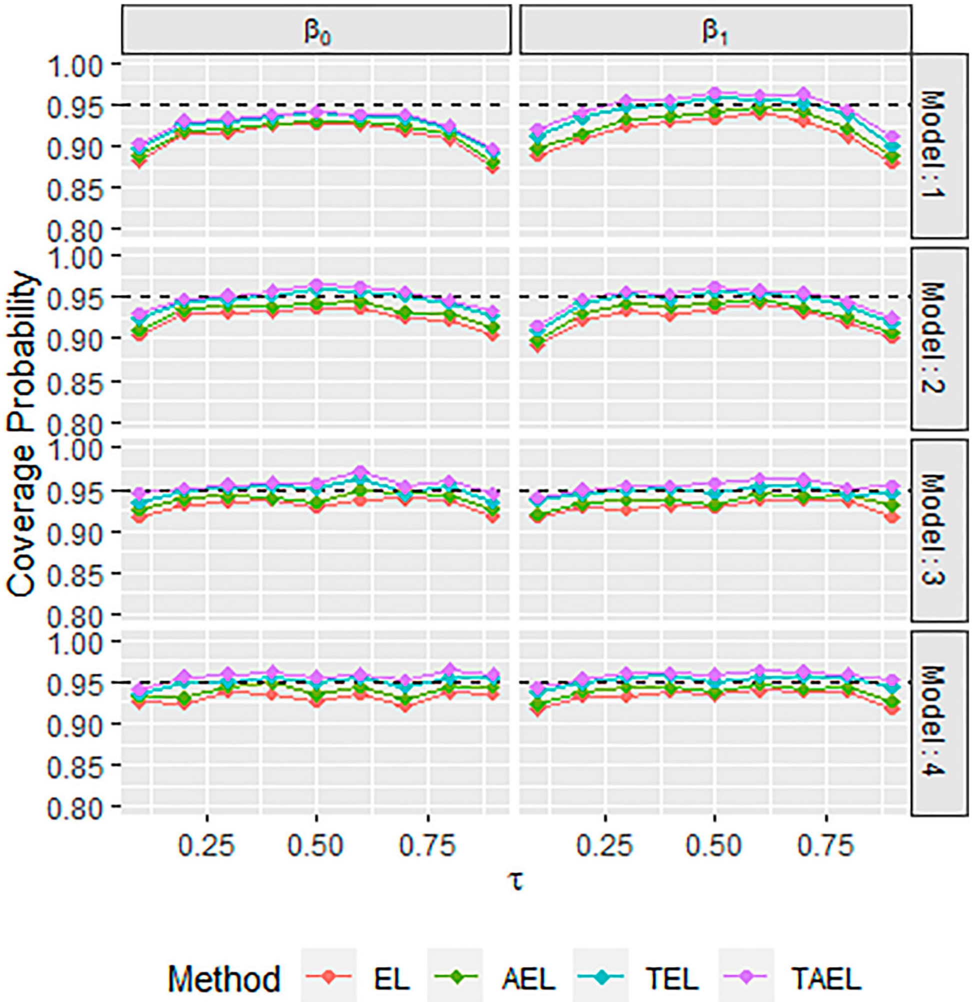

Figure 1. Coverage probabilities of the EL, AEL, TEL and TAEL methods with a range of τ values for sample size $n = 30$.

where $l^*(\cdot)$ defined in (12). Thus, for $\gamma = 1/2$, the transformed empirical log-likelihood ratio $l_t^*(\beta)$ can be defined as,

$$Z_t^*(l^*(\beta), \gamma) = l^*(\beta) \times \max\{1 - l^*(\beta)/n, 1/2\}. \tag{28}$$

More explicitly,

$$l_t^*(\beta) = \begin{cases} l^*(\beta)[1 - l^*(\beta)/n] & \text{if } l^*(\beta) \leq n/2, \\ l^*(\beta)/2 & \text{if } l^*(\beta) \geq n/2. \end{cases} \tag{29}$$

Theorem 3.3: *Let β_0 be the true parameter that satisfies $E\{Z_i(\beta)\} = 0$ and the $\Sigma(\beta_0) = E\{X_i^\top \psi_i(\beta_0)\psi_i(\beta_0)^\top X_i\}, i = 1, \ldots, n$, is of full rank. Let $l_t^*(\beta)$ be the adjusted profile log-likelihood ratio function defined in (29) and $a_n = o_p(n^{2/3})$. As $n \longrightarrow \infty$, we have*

$$-2l_t^*(\beta_0) \longrightarrow \chi^2(p)$$

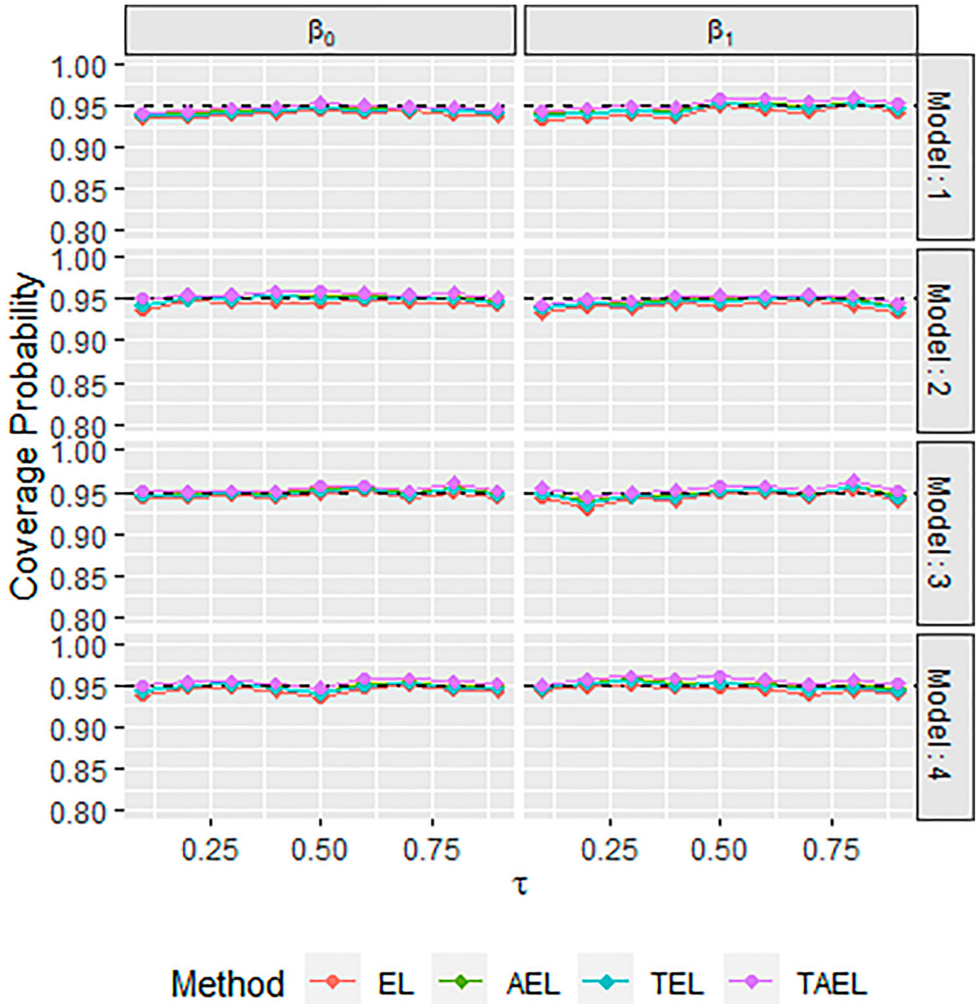

Figure 2. Coverage probabilities of the EL, AEL, TEL and TAEL methods with a range of τ values for sample size $n = 100$.

in distribution.

Proof: In order to proof the Theorem 3.3 we will follow the same strategy used in Theorem 3.2. Thus, details are omitted to conserve space. ■

3.2. Confidence regions

Theorem 3.4: *If Assumptions A1–A3 hold,*

(1) $-2l(\beta_0) \longrightarrow \chi^2(p)$
(2) $-2l^*(\beta_0) \longrightarrow \chi^2(p)$
(3) $-2l_t(\beta_0) \longrightarrow \chi^2(p)$
(4) $-2l_t^*(\beta_0) \longrightarrow \chi^2(p)$

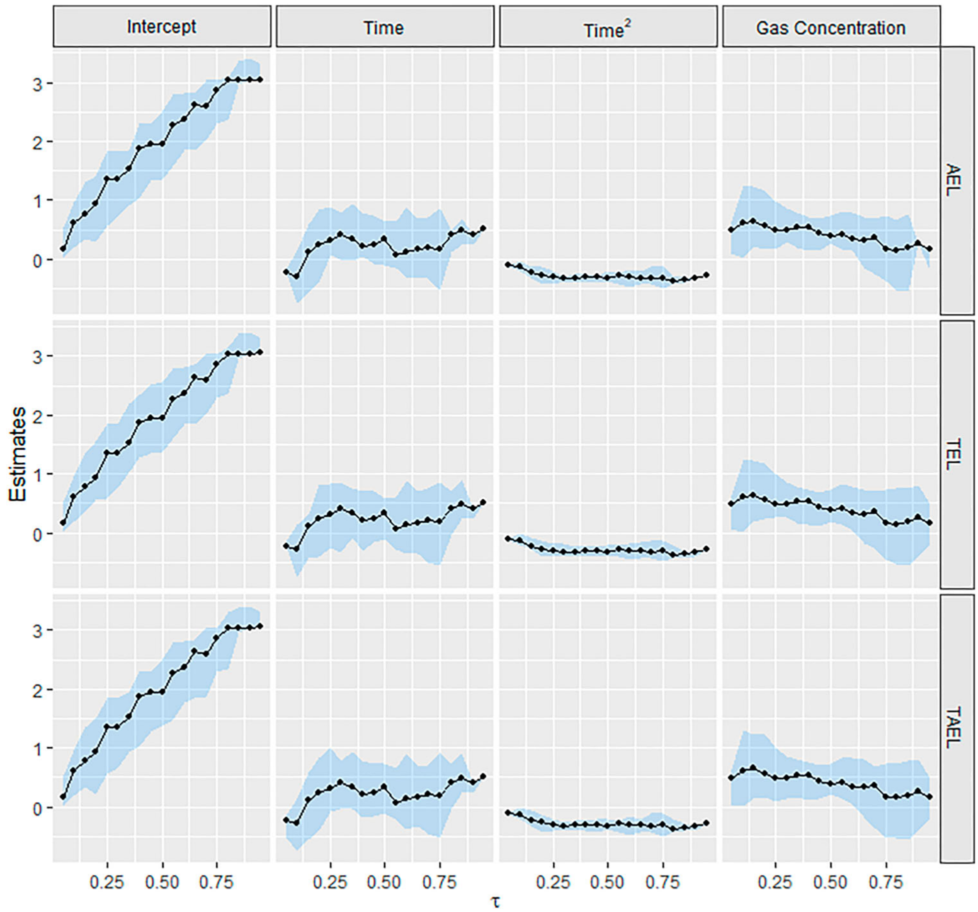

Figure 3. The point estimates (closed circles) and 95% pointwise confidence band of the quantile coefficients using AEL, TEL and TAEL.

Proof: The proof of Theorem 3.4 is similar to the proof of Theorem 2 given in [14]. Thus, details are omitted to conserve space. ■

As an analogy to parametric likelihoods, Theorem 3.4 allows us to use the test statistics $l(\beta_0), l^*(\beta_0), l_t(\beta_0)$ and $l_t^*(\beta_0)$ for testing or obtaining confidence regions for β_0. Specifically, we define

$$\begin{aligned} I_{EL} &= \{\beta : -2l(\beta) \leq \chi^2_{1-\alpha}(p)\}, \\ I_{AEL} &= \{\beta : -2l^*(\beta) \leq \chi^2_{1-\alpha}(p)\}, \\ I_{TEL} &= \{\beta : -2l_t(\beta) \leq \chi^2_{1-\alpha}(p)\}, \\ I_{TAEL} &= \{\beta : -2l_t^*(\beta) \leq \chi^2_{1-\alpha}(p)\}. \end{aligned} \tag{30}$$

as the EL, AEL, TEL and TAEL confidence regions for β_0, respectively, where $\chi^2_{1-\alpha}(p)$ is the $(1-\alpha)$th quantile of $\chi^2(p)$.

4. Simulation study

In this section, we conduct simulation studies to evaluate the performance of the proposed AEL, TEL and TAEL-based confidence regions for β with the existing EL and SEL based confidence intervals in terms of coverage probabilities and average lengths. We consider the same settings used in [14] such as homoscedastic and heteroscedastic error distributions. In addition, we use two additional settings including heavy-tailed and skewed error distributions.

- Model 1: (Homoscedastic)

$$y_{ij} = \beta_0 + x_{ij}\beta_1 + e_{ij}(\tau), \quad i = 1, \dots, n, \quad j = 1, \dots, N.$$

where $x_{ij} \sim N(0.5j, 0.5^2)$, and $(e_{i1}, \dots, e_{iN})^\top \sim N(0, V)$, where V has an exchangeable structure with diagonal entries 1 and off-diagonal entries 0.7, $e_{ij}(\tau) = e_{ij} - \Phi^{-1}(\tau)$ with Φ being the cumulative distribution function of $N(0, 1)$. Here $\Phi^{-1}(\tau)$ is subtracted from e_{ij} so that the τth quantile of $e_{ij}(\tau)$ is zero.

- Model 2: (Heteroscedastic)

$$y_{ij} = \beta_0 + x_{ij}\beta_1 + 0.25(1 + |x_{ij}|)e_{ij}(\tau), \quad i = 1, \dots, n, \quad j = 1, \dots, N.$$

where $x_{ij} \sim N(0.5j, 0.5^2)$, and $(e_{i1}, \dots, e_{iN})^\top \sim N(0, V)$, where V has an AR(1) correlation structure, i.e. $corr(e_{ij}, e_{ik}) = 0.7^{|j-k|}$, and $e_{ij}(\tau) = e_{ij} - \Phi^{-1}(\tau)$.

- Model 3: (Heavy-tailed)

$$y_{ij} = \beta_0 + x_{ij}\beta_1 + e_{ij}, \quad i = 1, \dots, n, \quad j = 1, \dots, N.$$

where $x_{ij} \sim N(0.1j, 1.75^2)$, and $(e_{i1}, \dots, e_{iN})^\top \sim \text{Cauchy}(0, 1)$,

- Model 4: (Skewed)

$$y_{ij} = \beta_0 + x_{ij}\beta_1 + e_{ij}, \quad i = 1, \dots, n, \quad j = 1, \dots, N.$$

where $x_{ij} \sim \text{Unif}(0, 1)$, and $(e_{i1}, \dots, e_{iN})^\top \sim \text{SN}(1, 0.5, 0.5)$. The probability distribution function of a skew normal random variable X is given by

$$f_X(x) = \frac{2}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\lambda\frac{x-\mu}{\sigma}\right), \quad x \in \mathbb{R}$$

where ϕ and Φ are the probability distribution function and cumulative distribution function of the standard normal distribution. We denote $X \sim SN(\mu, \sigma, \lambda)$. In all four models, we let $m = 10$, and $\beta_0 = \beta_1 = 1$. In our simulation study, we consider different quantile levels of interest, including $\tau = 0.5$ and 0.7 and various sample sizes $n = 30$, 50 and 100. The results are based on 2000 iterations. The results are summarized in Tables 1–4. In most of the scenarios, We observe that the AEL, TEL and TAEL based confidence intervals for

Table 5. Estimation and confidence intervals of quantile coefficients in the ophthalmology study at quantile level, $\tau = 0.25, 0.5, 0.75$.

τ	Variable	Estimate	95% Confidence Interval							
			EL	Length	AEL	Length	TEL	Length	TAEL	Length
0.25	Intercept	1.351	(0.759, 1.846)	1.087	(0.560, 1.839)	1.279	(0.588, 1.846)	1.258	(0.560, 1.839)	1.279
	Time (log t)	0.304	**(−0.151, 0.825)**	0.976	(0.075, 0.893)	0.818	**(−0.251, 0.827)**	1.078	**(−0.076, 1.003)**	1.079
	Time^2 (log^2 t)	−0.307	(−0.399, −0.186)	0.213	(−0.417, −0.211)	0.206	(−0.399, −0.172)	0.227	(−0.417, −0.211)	0.206
	Gas (x)	0.482	(0.263, 0.997)	0.734	(0.176, 0.992)	0.816	(0.257, 0.997)	0.740	(0.083, 0.992)	0.909
0.5	Intercept	1.944	(1.466, 2.523)	1.057	(1.365, 2.523)	1.158	(1.367, 2.572)	1.205	(1.365, 2.523)	1.158
	Time (log t)	0.340	**(−0.105, 0.592)**	0.697	**(−0.105, 0.638)**	0.743	**(−0.105, 0.593)**	0.698	**(−0.105, 0.737)**	0.842
	Time^2 (log^2 t)	−0.333	(−0.381, −0.203)	0.178	(−0.373, −0.205)	0.168	(−0.383, −0.203)	0.180	(−0.382, −0.205)	0.177
	Gas (x)	0.381	(0.274, 0.744)	0.470	(0.264, 0.740)	0.476	(0.199, 0.744)	0.545	(0.200, 0.740)	0.540
0.75	Intercept	2.868	(2.363, 3.045)	0.682	(2.303, 3.045)	0.742	(2.303 3.045)	0.742	(2.303, 3.045)	0.742
	Time (log t)	0.178	**(−0.422, 0.809)**	1.231	**(−0.518, 0.868)**	1.386	**(−0.541, 0.851)**	1.392	**(−0.542, 0.923)**	1.465
	Time^2 (log^2 t)	−0.315	(−0.479, −0.152)	0.327	(−0.490, −0.101)	0.389	(−0.484, −0.131)	0.353	(−0.493, −0.101)	0.392
	Gas (x)	0.171	**(−0.418, 0.742)**	1.160	**(−0.378, 0.740)**	1.118	**(−0.459, 0.742)**	1.201	**(−0.538, 0.740)**	1.278

the regression parameters have higher coverage probabilities than the EL and SEL based confidence regions. In particular, the TAEL based confidence regions gives better coverage for various sample sizes and at different quantile level. The EL-based confidence regions typically perform the worst of the methods overall, however in some cases slightly better than the SEL. On the other hand, the SEL method has the second-worst coverage rates. Especially, in Model 4, when the sample size is small, coverage probabilities of EL and SEL are lower than the nominal level. However, the TEL and TAEL have coverage probabilities that are closer to the nominal confidence level. Not surprisingly, as sample size increases, all five methods give better coverage probabilities, however, coverage probabilities of AEL, TEL and TAEL are much better than the existing methods. Generally, the confidence regions of the AEL, TEL, and TAEL have longer average lengths than those of EL and SEL, however, they are still in an acceptable range. Our simulation results suggest, in some cases, the TAEL has an over-coverage problem, although coverage probabilities are slightly higher than the nominal level with only around 0.96. In those cases, we recommend to use AEL or TEL methods.

Next, we sketch the coverage probabilities of the EL, AEL, TEL and TAEL based confidence regions for sample sizes $n = 30$ and 100 and at various quantile levels. The results are shown in Figures 1 and 2. For a small sample size (for example, $n = 30$), when τ increases, the coverage probability increases at first to a maximum towards the middle, then decreases. Not surprisingly, when the sample size increases from 30 to 100, all methods give coverage probabilities close to the nominal level. Overall, for all quantile levels, the coverage probabilities of AEL, TEL and TAEL are higher than the EL method.

5. Real data analysis

In this section, we apply our proposed methods to demonstrate the effectiveness of AEL, TEL, and TAEL in constructing confidence intervals by analyzing an ophthalmology data set. This data set was used in [14,23]. Intraocular gas was pumped into the eyes of 31 patients before retinal repair operations to provide an internal retinal split tamponade. The follow-up of patients was performed 3 to 8 times in three months after the operation, and gas leftovers were calculated as a proportion of the original gas content in the eyes. Wang and Zhu [14] studied how the conditional quantiles of gas decay with time. Similar to [14], we let y_{ij} be the gas volume left in the eye of patient i at day t_{ij}. We define the logit-transformed response

$$\hat{y}_{ij} = \log\left(\frac{y_{ij} + 0.05}{1 - y_{ij} + 0.05}\right),$$

where the constant 0.05 is added to avoid zero denominators. Let x_i be the centered gas concentration of the ith subject, so that $x_i = -1, 0, 1$ corresponding to gas concentration levels of 15, 20 and 25, respectively. We consider the following quantile regression model

$$\hat{y}_{ij} = \beta_0(\tau) + \beta_1(\tau)\log(t_{ij}) + \beta_2(\tau)\log^2(t_{ij}) + \beta_3(\tau)x_i + e_{ij}(\tau)$$

where the tth conditional quantile of $e_{ij}(\tau)$ given the other covariates is zero. In our analysis, we compute confidence intervals for the EL, AEL, TEL and TAEL at $\tau = 0.25, 0.5$ and 0.75. Table 5 summarizes the coefficient estimations and 95% confidence intervals for all four methods. We observe that the EL method provides the narrower confidence intervals while TAEL gives the longest. All four procedures agree in terms of the significance of effects at $\tau = 0.25, 0.5$ and 0.75. However, at $\tau = 0.25$, the EL, TEL and TAEL methods do not detect the time effects. In most cases, the average lengths of 95% AEL, TEL and TAEL confidence intervals are higher than that of the EL-based confidence interval. Figure 3 illustrates the point estimates (closed circles), and the shaded area depicts a 95% pointwise confidence band obtained from the AEL, TEL and TAEL methods.

6. Conclusion

In this paper, we study the modified empirical likelihood methods including the adjusted empirical likelihood (AEL), the transformed empirical likelihood (TEL), and the transformed adjusted empirical likelihood (TAEL) on constructing confidence intervals for quantile regression models with longitudinal data. The profile log-EL statistics under the true values of the parameters share the same asymptotic properties with the original EL. Simulations under various scenarios are conducted to compare the proposed procedures with the existing method proposed in [14] in terms of coverage probabilities and average lengths of the confidence intervals. The simulation results indicate that the proposed AEL, TEL, and TAEL provide better coverage probabilities which are closer to the nominal confidence level than the EL and SEL based coverage probabilities. A real data application is given to illustrate the construction of confidence intervals by the proposed methods.

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