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ORIGINAL ARTICLE

Some New Bounds for Moment Generating Functions of Various Life Distributions Using Mean Residual Life Functions

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Abstract
In this paper, we present bounds for moment generating functions of certain life distributions, including the decreasing mean residual life class. We also give bounds for some other much larger classes of life distributions. Most bounds are based on the mean residual life functions. These bounds are compared to some other bounds in the literature. The methods used can also find bounds for other functions of the lifetime random variable. Some applications to bounding tail probabilities of lifetime distributions are given.

1 Introduction

Let X denote a non-negative (lifetime) random variable having continuous distribution function (d.f.) $F(x) = P(X \leq x)$ and survivor function $\bar{F}(x) = 1 - F(x)$, $x \geq 0$. Many notions of aging have been studied in the reliability literature. The interested reader may refer to Jeong [11] for further information. For this paper, the three most relevant are the increasing failure rate (IFR), the decreasing mean residual life (DMRL), and the new better than used in expectation (NBUE) classes of lifetime distributions.

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Definition 1.1

a. A lifetime distribution F is IFR if for all $x \geq 0$, $\dfrac{\bar{F}(x+t)}{\bar{F}(t)}$ is decreasing in $t \geq 0$. If $f = F^{'}$ is the probability density function of X, this is equivalent to $h(t) = \dfrac{f(t)}{\bar{F}(t)}$ being a non-decreasing function of $t \geq 0$.
b. A lifetime distribution F is DMRL if

$$g(t) = E(X - t|X \geq t) = \frac{\int_t^\infty \bar{F}(w)\,\mathrm{d}w}{\bar{F}(t)} \tag{1}$$

is decreasing in $t \geq 0$.
c. A lifetime distribution F is NBUE if

$$\int_0^\infty \bar{F}(t+x)\,\mathrm{d}t \leq \mu \bar{F}(x) \quad \text{for all } x \geq 0, \tag{2}$$

where $\mu = E(X) = \int_0^\infty \bar{F}(x)\,\mathrm{d}x$.

It is known that the IFR class is contained in the DMRL class, which, in turn, is contained in the NBUE class. See Barlow and Proschan [3], for example. In particular, we shall be concerned with obtaining bounds for the moment generating function (mgf) of X given by

$$\phi(t) = M_X(t) = E(e^{tX}) = \int_0^\infty e^{tX}\,\mathrm{d}F(x). \tag{3}$$

We shall be mainly focused on values of $t \geq 0$, although we do obtain some bounds for $t < 0$ also. The reason for this is the famous Chernoff's inequality which is given below.

Definition 1.2 (*Chernoff's Inequality*) Let X be random variable and let a be a real number. Then, we have,

$$P(X \geq a) \leq \inf_{t \geq 0} \left[M_X(t) e^{-at} \right]. \tag{4}$$

Chernoff's inequality gives an upper bound on the tail probability given in (4) and is generally good as $a \longrightarrow \infty$. Clearly, replacing $M_X(t)$ by an upper bound in (4) still provides an upper bound on $P(X \geq a)$. Hence, the bounds derived in this paper for $t \geq 0$ will be useful for obtaining bounds on $P(X \geq a)$, especially for large values of a. Bound (4) is an example of a large deviation type of bound, and there is an extensive literature available on this topic. See, for example, Petrov [12].

In this paper, we shall use various methods to obtain upper (and a few lower) bounds on $M_X(t)$, particularly for the DMRL class of life distributions. Then, we shall obtain some bounds for some other classes of life distributions, some of which properly contain the DMRL class. The mean residual life function given by (1) will play a prominent role in the construction of most of the bounds presented. We shall then compare the new bounds to some other bounds that have been given in the literature. Previously, no bounds on $M_X(t)$ have been proposed specifically for the DMRL class. However, Ahmad and Mugdadi [1] have proposed bounds for the bigger NBUE class and other classes even larger than the NBUE class. Our bounds are neither more or less general than the bounds of Ahmad and Mugdadi [1], since we shall also consider life distribution classes that are not NBUE and not of any class considered by these authors. Our method will extend to bounding expected values of other functions of X besides the exponential function corresponding to the mgf. We shall also consider an application of our results to Chernoff's theorem for sums of DMRL random variables.

2 Preliminary Results

To derive all new bounds, we shall need some preliminary results. These are given next below. Lemma 2.1 can be obtained immediately from Theorem 1.4 in Farissi et al. [7].

Lemma 2.1 *Let $w(x)$ be a real-valued function defined on an interval I of real numbers. Suppose $w(x)$ is twice differentiable on I^0, the interior of I. Suppose $m = \inf_{x\in I^0} w''(x)$ and $M = \sup_{x\in I^0} w''(x)$ both exist. Let X be a random variable with mean μ and variance $\sigma^2 < \infty$. Then,*

$$w(\mu) + \frac{1}{2}m\sigma^2 \le E\big[w(X)\big] \le w(\mu) + \frac{1}{2}M\sigma^2. \tag{5}$$

Lemma 2.2 is given in [14, p. 92]. We shall only need the case where the random variable X is contained in some interval I of real numbers.

Lemma 2.2 *Let X have distribution function F. Let $w_1(x)$ and $w_2(x)$ be monotonic functions of x defined on an interval $[a, b]$, where $-\infty \le a < b \le \infty$.*

a. If $w_1(x)$ and $w_2(x)$ are either both non-decreasing or both non-increasing, then $E[w_1(X)w_2(X)] \ge E[w_1(X)]E[w_2(X)]$, that is,

$$\int_a^b w_1(x)w_2(x)\,\mathrm{d}F(x) \ge \int_a^b w_1(x)\,\mathrm{d}F(x) \int_a^b w_2(x)\,\mathrm{d}F(x), \tag{6}$$

provided all three integrals in (6) exist.

b. If $w_1(x)$ is non-decreasing and $w_2(x)$ is non-increasing, then the inequality signs are reversed above.

Note that Lemma 2.2 is a covariance result, since in part (a), $Cov[w_1(X), w_2(X)] \geq 0$, for example. Lemma 2.3 gives some moment inequality relations that will be needed later. See Barlow et al. [2], p. 394 or Fagiuoli and Pellerey [6].

Lemma 2.3 *Suppose X has a DMRL distribution. Let n and m be non-negative integers with* $1 \leq n \leq m$. *Then, we have*

$$\left[\frac{E(X^n)}{n!}\right]^{1/n} \geq \left[\frac{E(X^m)}{m!}\right]^{1/m}. \tag{7}$$

In particular, when $n = 1$,

$$E(X^m) \leq m!\mu^m. \tag{8}$$

When $n = 1$ *and* $m = 2$, $E(X^2) \leq 2\mu^2$, *which gives,*

$$\sigma^2 \leq \mu^2. \tag{9}$$

Lemma 2.4 gives an infinite series representation for $E[w(X)]$ in terms of the mean residual life function $g(x)$. See from [9], pp. 25–26.

Lemma 2.4 *Let X be a continuous lifetime random variable with support* $[a, b)$ *where* $0 \leq a < b \leq \infty$. *Let* $w(x)$ *be a function which is real analytic on some open interval containing* $[a, b)$. *Let* $\mu = E(X)$. *Then,*

$$E[w(X)] = w(\mu) + \sum_{j=2}^{\infty} \left(\frac{j-1}{j!}\right) \int_0^{\infty} w^{(j)}(x)(g(x))^j \,\mathrm{d}F(x), \tag{10}$$

where $w^{(j)}(x)$ *is the jth derivative of w,* $j = 2, 3, \ldots$ *and the function* $g(x)$ *is defined in Section 1 in equation (1) for the DMRL class.*

In particular, if $w(x) = x^2$, then the variance of X is

$$\sigma^2 = Var(X) = E(X^2) - \mu^2 = \int_0^{\infty} (g(x))^2 \,\mathrm{d}F(x). \tag{11}$$

Next, we present the two most relevant bounds from the many bounds on $M_X(t)$ given in Ahmad and Mugdadi [1].

Theorem 2.1 (Theorem 2.1 of Ahmad and Mugdadi [1]) *Suppose X has an NBUE lifetime distribution with* $\mu = E(X) < \infty$. *Then,*

$$\begin{aligned} M_X(t) =& E(e^{tX}) \leq \frac{1}{1-\mu t}, \quad 0 \leq t < \frac{1}{\mu}, \\ \equiv& M_{A_1}(t). \end{aligned} \tag{12}$$

Theorem 2.2 (*Theorem* 2.2 *of Ahmad and Mugdadi* [1]) *Suppose X has a new better than renewal used in expectation* (*NBRUE*) *lifetime distribution*, *that is*,

$$\int_x^\infty \left(\int_u^\infty \bar{F}(w)\,\mathrm{d}w \right) \mathrm{d}u \le \mu \int_x^\infty \bar{F}(u)\,\mathrm{d}u.$$

Suppose $E(X^2) = \mu_{(2)} < \infty$. *Then*,

$$\begin{aligned} M_X(t) \le & \frac{2 + \mu_{(2)}t^2 - 2\mu^2 t^2}{2(1 - \mu t)}, \quad 0 \le t < \frac{1}{\mu}. \\ \equiv & M_{A_2}(t). \end{aligned} \tag{13}$$

In particular, (13) holds for the NBUE and DMRL classes which are contained in the NBRUE class. By Eq. (12), we see that $M_{A_2}(t) \le M_{A_1}(t)$, for $0 \le t < \frac{1}{\mu}$.

Remark 1 No bound on $M_X(t)$ has been previously proposed which uses knowledge of both μ and $\sigma^2 = \mu_{(2)} - \mu^2$ specifically for the DMRL class. But Theorem 2.2 is one of the few bounds in the literature utilizing both μ and σ^2. Also, $M_{A_2}(t)$ is not an upper bound for $M_X(t)$ if $t < 0$. It is a lower bound instead. We shall compare $M_{A_1}(t)$ and $M_{A_2}(t)$ to the new upper bound $M_1(t)$ given in Sect. 3, in the DMRL case.

Of course, the $M_{A_1}(t)$ and $M_{A_2}(t)$ bounds are for a more general class of life distributions. But many commonly used lifetime distributions, such as the Weibull and Gamma distributions, are members of the DMRL, NBUE, and NBRUE classes. Bounds based on the mean μ alone are not enough to discriminate between these classes since the $M_{A_1}(t)$ bound is sharp for all of these classes, if only the mean μ is to be used. This is because the exponential distribution is a common boundary distribution that is a member of all these classes. However, if $\sigma^2 < \mu^2$ holds (true for all non-exponential DMRL lifetime distributions), then the exponential distribution is 'thrown' out and we can obtain smaller bounds on $M_X(t)$. Moreover, the methods used to obtain bounds in the DMRL case will sometimes work for larger classes of distributions which are not of the NBUE or NBRUE classes, as we shall see later in Sect. 3.

We shall present some examples of this when we are done with presenting results for the DMRL class in Sect. 3. Our results are meant to complement those of Ahmad and Mugdadi [1], and numerical comparisons are done just to get an idea of how much improvement in bounds is possible if we assume the more restrictive DMRL class as opposed to the NBUE and NBRUE classes. It should be mentioned that bounds for the mgf $M_X(t)$ have been given for distributions of bounded support in Brook [4], but we are often interested in distributions with infinite support in reliability theory.

3 Main Results

In this section, we present new bounds for the mgf $M_X(t)$ for various classes of life distributions. First, we consider bounds for the DMRL class. The first few bounds utilize both the mean μ and the variance σ^2 of X and provide the best bounds given in this paper. Theorem 3.1 gives a third upper bound on $M_X(t)$ utilizing both the mean μ and variance σ^2 of X.

Theorem 3.1 *Let X have a DMRL lifetime distribution with mean μ and variance σ^2. Let t_1 be the unique positive real number such that $\left(\frac{\sigma}{\mu}\right)^2\left[1+(\mu t_1-1)e^{\mu t_1}\right]=1$. Then, for $0 \le t < t_1$,*

$$\begin{aligned} M_X(t) \le & \frac{e^{\mu t}}{1-\left(\frac{\sigma}{\mu}\right)^2\left((\mu t-1)e^{\mu t}+1\right)} \\ \equiv & M_1(t). \end{aligned} \tag{14}$$

Proof We shall apply Lemma 2.4 with $w(x)=e^{tx},\ t \ge 0$. Then,

$$\begin{aligned} M_X(t)-e^{t\mu} &= \sum_{j=2}^{\infty}\left(\frac{j-1}{j!}\right)\int_0^\infty w^{(j)}(x)(g(x))^j \,\mathrm{d}F(x) \\ &= \sum_{j=2}^{\infty}\left(\frac{j-1}{j!}\right)\int_0^\infty t^j e^{tx}(g(x))^{j-2}(g(x))^2 \,\mathrm{d}F(x). \end{aligned}$$

Since X is DMRL, $g(x) \le g(0)=\mu,\ x \ge 0$. So

$$M_X(t)-e^{t\mu} \le \sum_{j=2}^{\infty}\left(\frac{j-1}{j!}\right)t^j\mu^{j-2}\int_0^\infty e^{tx}(g(x))^2 \,\mathrm{d}F(x). \tag{15}$$

Apply Lemma 2.2 with $w_1(x)=e^{tx}, w_2(x)=(g(x))^2$. Since w_1 is non-decreasing and w_2 is non-increasing in x,

$$\begin{aligned} \int_0^\infty e^{tx}(g(x))^2 \,\mathrm{d}F(x) \le & \int_0^\infty e^{tx} \,\mathrm{d}F(x)\int_0^\infty (g(x))^2 \,\mathrm{d}F(x) \\ = & M_X(t)\sigma^2. \end{aligned} \tag{16}$$

From (15) and (16), we obtain, using simple Maclaurin series expansions,

$$\begin{aligned} M_X(t)-e^{t\mu} \le & \left(\sum_{j=2}^{\infty}\left(\frac{j-1}{j!}\right)(\mu t)^j\right)M_X(t)\left(\frac{\sigma}{\mu}\right)^2 \\ = & \left(1+(\mu t-1)e^{t\mu}\right)M_X(t)\left(\frac{\sigma}{\mu}\right)^2. \end{aligned} \tag{17}$$

Solving for $M_X(t)$ in (17), we obtain

$$M_X(t) \leq \frac{e^{t\mu}}{1 - \left(\frac{\sigma}{\mu}\right)^2 \left((\mu t - 1)e^{t\mu} + 1\right)} \equiv M_1(t). \tag{18}$$

for all t with $0 \leq t < t_1$. This completes the proof of Theorem 3.1. □

Remark 2 The value of t_1 in Theorem 3.1 exceeds $\frac{1}{\mu}$ as we now demonstrate, unless $\sigma = \mu$ (the exponential distribution case). Let $R(x) = (x-1)e^x + 1$, where $x = \mu t$. It suffices to show $R(x) \leq 1,\ x \leq 0 \leq 1$, since $\left(\frac{\sigma}{\mu}\right)^2 \leq 1$ with $\left(\frac{\sigma}{\mu}\right)^2 < 1$, unless X is DMRL exponential. Now $R(0) = 0 \leq 1$ and $R(1) = 1 \leq 1$. We obtain $R'(x) = xe^x > 0$. So $R(x)$ is increasing in x, $0 \leq x \leq 1$. Thus, $R(x) \leq 1$ and $\left(\frac{\sigma}{\mu}\right)^2 \left((\mu t - 1)e^{t\mu} + 1\right) < 1$, unless $\sigma = \mu$ in the exponential distribution case. In addition, if $\sigma < \mu$, then

$$M_1(t) = \frac{e^{t\mu}}{1 - \left(\frac{\sigma}{\mu}\right)^2 \left((\mu t - 1)e^{t\mu} + 1\right)} \leq \frac{e^{t\mu}}{1 - \left((\mu t - 1)e^{t\mu} + 1\right)} = \frac{1}{1 - \mu t}, \quad 0 \leq t < \frac{1}{\mu}.$$

Thus, $M_1(t)$ is a smaller bound than $M_{A_1}(t)$, unless $\sigma = \mu$ in the exponential case.

Theorem 3.1 offers a new bound on the mgf of X when X has a DMRL lifetime distribution with known mean μ and variance σ^2. How does it compare to the Ahmad and Mugdadi [1] bound $M_{A_2}(t)$ which also uses the mean and variance? Of course, the upper bound $M_{A_2}(t)$ is for a more general class of distributions, so comparisons given below may be a bit delicate. However, a comparison of these bounds would give one an idea of the relative merit of the new bound.

Table 1 Comparison of upper bounds for $M_X(t)$ for Weibull distribution

θ	t	$M_X(t)$	$M_{A_1}(t)$	$M_{A_2}(t)$	$M_1(t)$	t_0	t_1
1.1	0.35	1.485	1.510	1.495	1.490	1.036	1.11
1.5	0.35	1.406	1.462	1.423	1.412	1.108	1.456
2.0	0.35	1.382	1.450	1.399	1.386	1.128	1.759
3.0	0.45	1.511	1.672	1.555	1.516	1.120	2.175
5.0	0.40	1.449	1.580	1.479	1.450	1.089	2.715

We compare various upper bounds on $M_X(t)$ for the Weibull and Gamma distributions with probability density functions given below. Table 1 presents various upper bounds for $M_X(t)$ as well as the value of t_1 defining the interval of existence (of upper bound). The quantity t_0 is the quantity $\frac{1}{\mu}$ giving the interval of existence of $M_{A_1}(t)$ and $M_{A_2}(t)$.

1. Weibull distribution. X has probability density function

$$f(x) = \theta x^{\theta-1} e^{-x^{\theta}}, \quad x > 0,\ \theta > 0$$

   X has a DMRL distribution iff $\theta \geq 1$.
2. Gamma distribution. X has probability density function

$$f(x) = \frac{1}{\Gamma(\theta)} x^{\theta-1} e^{-x}, \quad x > 0,\ \theta > 0$$

   X has a DMRL distribution iff $\theta \geq 1$.

Tables 1 and 2 provide a comparison of upper bounds for $M_X(t)$. For both the Weibull and Gamma distributions, we considered the set of θ values: $\theta \in \{1.1, 1.5, 2.0, 3.0, 5.0\}$. It can be clearly seen that $M_1(t)$ is the best upper bound.

Next, we present bounds for $M_X(t)$ for some lifetime distribution classes that are not DMRL. In some cases, we can obtain bounds for the IMRL class (increasing mean residual life class). Generally, the bounds of Theorem 3.2 assumes $g(x)$ is bounded on $[0, \infty)$, but makes no assumptions on monotonicity of $g(x)$. Thus, Theorem 3.2 is still applicable to the DMRL class, since $0 \leq g(x) \leq g(0) = \mu$ for the DMRL class. However, the bounds given below in Theorem 3.2 produce inferior bounds when compared to the more specific bounds of Theorem 3.1.

Theorem 3.2 *Suppose X has a lifetime distribution with bounded mean residual life function. Suppose there is a real number M such that* $0 \leq g(x) \leq M,\ x \geq 0$. *Then, if* $\mu = E(X)$, *then*

Table 2 Comparison of upper bounds for $M_X(t)$ for Gamma distribution

θ	t	$M_X(t)$	$M_{A_1}(t)$	$M_{A_2}(t)$	$M_1(t)$	t_0	t_1
1.1	0.35	1.606	1.626	1.615	1.610	0.909	0.941
1.5	0.25	1.540	1.600	1.562	1.549	0.667	0.771
2.0	0.25	1.778	2.000	1.875	1.807	0.500	0.639
3.0	0.18	1.814	2.174	1.963	1.846	0.333	0.488
5.0	0.10	1.694	2.000	1.800	1.709	0.200	0.344

$$
\begin{aligned}
M_X(t) \leq & \frac{e^{-t(M-\mu)}}{1-Mt} \\
\equiv & M_2(t), \quad 0 \leq t < t_2,
\end{aligned} \tag{19}
$$

where $t_2 = \dfrac{1}{M}$.

Proof Proceed as in the proof of Theorem 3.1. Then, we obtain

$$
\begin{aligned}
M_X(t) - e^{t\mu} &= \sum_{j=2}^{\infty} \left(\frac{j-1}{j!}\right) \int_0^{\infty} t^j e^{tx} (g(x))^j \, \mathrm{d}F(x) \\
&\leq \sum_{j=2}^{\infty} \left(\frac{j-1}{j!}\right) \int_0^{\infty} t^j e^{tx} M^j \, \mathrm{d}F(x) \\
&= \left(\sum_{j=2}^{\infty} \left(\frac{j-1}{j!}\right) (Mt)^j \right) M_X(t) \\
&= \left(1 + (Mt-1)e^{Mt} \right) M_X(t).
\end{aligned} \tag{20}
$$

Thus,

$$
M_X(t) - e^{t\mu} \leq M_X(t) + \left((Mt-1)e^{Mt} \right) M_X(t),
$$

which gives,

$$
-e^{t\mu} \leq \left((Mt-1)e^{Mt} \right) M_X(t).
$$

Since $Mt - 1 < 0$, we obtain

$$
\begin{aligned}
M_X(t) \leq & \frac{e^{t\mu}}{(1-Mt)e^{Mt}} \\
= & \frac{e^{-t(M-\mu)}}{1-Mt}, \quad 0 \leq t < \frac{1}{M}.
\end{aligned} \tag{21}
$$

This completes the proof. □

Table 3 Comparison of upper bounds for $M_X(t)$ for Weibull distribution

θ	t	$M_X(t)$	$M_2(t)$	M	t_2
1.1	0.35	1.485	2.257	1.965	0.509
1.5	0.35	1.406	2.110	1.903	0.525
2.0	0.35	1.382	2.073	1.886	0.530
3.0	0.45	1.511	4.304	1.893	0.528
5.0	0.40	1.449	2.880	1.918	0.521

Table 4 Comparison of upper bounds for $M_X(t)$ for Gamma distribution

θ	t	$M_X(t)$	$M_2(t)$	M	t_2
1.1	0.35	1.606	2.659	2.100	0.476
1.5	0.25	1.540	2.077	2.500	0.400
2.0	0.25	1.778	3.115	3.000	0.333
3.0	0.18	1.814	2.983	4.000	0.250
5.0	0.10	1.694	2.262	6.000	0.167

It is clearly evident that the bound presented in Theorem 3.2 depends on the value of M, which is the known upper bound of the MRL function. In the special case of DMRL life distributions, a valid choice of M is $M = \mu = g(0)$. Therefore, for any valid choice of M, in the special case of DMRL distributions, the bound of Theorem 3.2 is inferior to that of Theorem 3.1. However, the bound of Theorem 3.2 may be useful in the DMRL case, when one knows μ and M, but does not know $\mu_{(2)}$ or σ^2. Tables 3 and 4 present Theorem 3.2 bounds for the Weibull and Gamma distributions. Comparing to the more specific bounds of Tables 1 and 2, the bound $M_2(t)$ is not as good as $M_1(t)$ as expected. However, $M_2(t)$ is more generally applicable.

Remark 3 Let $m = \inf\{g(t) : t \geq 0\}$. If $m > 0$, then slightly modifying the proof of Theorem 3.2, we can obtain the lower bound,

$$M_X(t) \geq \frac{e^{-t(m-\mu)}}{1-mt}, \quad 0 \leq t < \frac{1}{m}. \tag{22}$$

Example 1 Suppose X has survivor function

$$\bar{F}(x) = S(x) = \frac{x+1}{(2x+1)^{5/4}} e^{-x/2}, \quad x \geq 0.$$

Then, a simple computation gives $g(x) = \dfrac{2x+1}{x+1}$, with $1 \leq g(x) \leq 2$. Letting $M = 2$ in Theorem 3.2 gives

$$M_X(t) \leq \frac{e^{-t}}{1-2t}, \quad 0 \leq t < \frac{1}{2}. \tag{23}$$

Note that X has a bounded increasing mean residual life function (IMRL), and is not NBUE or NBRUE. Hence, only the bound (23) is applicable (Table 5). Theorem 3.3 gives a simple condition one can check to see if $g(x)$

Table 5 Bound (23) values

t	0.10	0.20	0.30	0.40	0.45
$M_X(t)$	1.094	1.199	1.318	1.451	1.524
Bound (23)	1.131	1.365	1.852	3.352	6.376

is bounded above by using the hazard (failure rate) function. In the sequel, let $S(x) = \bar{F}(x) = 1 - F(x),\ x \geq 0$.

Theorem 3.3 *Suppose that the hazard function* $h(x) = \dfrac{f(x)}{S(x)}$ *exists and satisfies* $\lim_{x\to\infty} h(x) = \infty$. *Suppose that* $f(x)$ *is continuous on* $[0, \infty)$. *Then,* $g(x)$ *is bounded on* $[0, \infty)$, *that is, there exists* $M > 0$ *such that* $0 \leq g(x) \leq M,\ x \geq 0$.

Proof Since $f(x)$ is continuous, $f(0)$ exists and $g(x)$ is continuous on $[0, \infty)$. To show $g(x)$ is bounded above, it now suffices to show $\lim_{x\to\infty} g(x) = 0$, and we obtain using L'hospital's rule,

$$\begin{aligned}
\lim_{x\to\infty} g(x) &= \lim_{x\to\infty} \frac{\int_x^\infty S(w)\,\mathrm{d}w}{S(x)} \\
&= \lim_{x\to\infty} \frac{S(x)}{f(x)} \\
&= \frac{1}{\lim_{x\to\infty} h(x)} \\
&= 0.
\end{aligned}$$

This completes the proof. □

Remark 4 In the proof of Theorem 3.3, we used the relationship between $g(x)$ and $h(x)$ as $x \to \infty$. More precise estimations of $g(t)$ can be obtained if $\lim_{x\to\infty} h(x) = \infty$. See Theorem 1 of Gupta and Bradley [10]. Also, Theorem 3.3 is also true if $\lim_{x\to\infty} h(x) = L$, provided $L \neq 0$ $(L > 0)$.

Next, we present bounds for $M_X(t)$ for a very general class of lifetime distributions. The bounds of Theorem 3.4 are valid, in particular, for lifetime distributions which are not NBUE (hence not DMRL), provided the mean and variance of X exist.

Theorem 3.4 *Suppose* X *has a lifetime distribution with mean* μ *and variance* $\sigma^2 < \infty$. *Then,*

$$M_X(t) \leq e^{t\mu} + \frac{1}{2}t^2\sigma^2, \quad t < 0, \tag{24}$$

and

$$M_X(t) \geq e^{t\mu} + \frac{1}{2}t^2\sigma^2, \quad t \geq 0. \tag{25}$$

Proof Apply Lemma 2.1 with $w(x) = e^{tx}$. Then, $w''(x) = t^2 e^{tx}$. If $t < 0$, $M = \sup_{x>0} t^2 e^{tx} = t^2$. If $t \geq 0$, $m = \inf_{x>0} t^2 e^{tx} = t^2$. □

We have obtained upper bounds for $M_X(t)$ for $t < 0$ also, but these are not as good as the upper bounds of Ahmad and Mugdadi [1] for this case (when excluding their Theorem 2.2 and Theorem 3.2 bounds which are not valid for $t < 0$).

4 Bounds on Derivatives of mgfs

In this section, we show how bounds on derivatives of mgfs may be obtained. Ahmad and Mugdadi [1] obtained results on bounds for $\Psi(t) = tM'_X(t)$ for various reliability classes of life distributions. See their Theorems 3.1–3.4 for the NBU, NBRU, RNBU, and RNBRU classes of life distributions, respectively. (In Theorems 3.2 and 3.3, t must be non-negative.)

Theorem 4.1 is just one bound of many that can be derived for $M'_X(t)$ using the methods of this paper. Bounds for higher-order derivatives $M_X(t)$ can also be obtained.

Theorem 4.1 *Suppose X has a DMRL lifetime distribution with mean μ and variance* σ^2. *Then,*

$$M'_X(t) \le \mu + (\mu^2 + \sigma^2)t + \frac{3\left(\frac{t}{t_3}\right)^3 - 2\left(\frac{t}{t_3}\right)^4}{\left(1 - \frac{t}{t_3}\right)^2}, \quad 0 \le t < t_3, \tag{26}$$

where $t_3 = \sqrt{\frac{E(X^2)}{2}} = \sqrt{\frac{\mu^2 + \sigma^2}{2}}$.

Proof We have

$$M'_X(t) = \mu + (\mu^2 + \sigma^2)t + \sum_{m=3}^{\infty} \frac{E(X^m)}{m!} t^m. \tag{27}$$

From (7) of Lemma 2.3 with $n = 2$,

$$E(X^m) \le \frac{m!}{2^{m/2}} (E[X^2])^{m/2}, \quad m = 3, 4, \ldots$$

From (27), we obtain

$$\begin{aligned} M'_X(t) &\le \mu + (\mu^2 + \sigma^2)t + \sum_{m=3}^{\infty} m\left(\frac{t}{t_3}\right)^m \\ &= \mu + (\mu^2 + \sigma^2)t + \frac{3\left(\frac{t}{t_3}\right)^3 - 2\left(\frac{t}{t_3}\right)^4}{\left(1 - \frac{t}{t_3}\right)^2}, \quad 0 \le t < t_3. \end{aligned}$$

□

Table 6 Comparison of upper bounds for Weibull distribution

θ	σ/μ	a	t	Bound (28)	Bound (30)
1.2	0.8366	2.9410	0.8606	0.2985	0.1194
2.0	0.5227	2.8860	1.3893	0.1217	0.1046
3.0	0.3634	2.8930	1.7966	0.0584	0.1065
5.0	0.2291	2.9180	2.3317	0.0215	0.1132

5 Applications

As mentioned earlier, one application area is to use an upper bound on the mgf in Chernoff's inequality. The best bound in the DMRL case would be $M_1(t)$. Thus, for $a > 0$,

$$P(X \geq a) \leq \inf_{t \geq 0} M_1(t) e^{-at}. \tag{28}$$

Cheng and He [5], it was shown that if X has a DMRL distribution with $E(X^2) < \infty$, then

$$\sup_{t \geq 0} |\bar{F}(t) - e^{-t/\mu}| \leq \frac{1}{2}\left(1 - \left(\frac{\sigma}{\mu}\right)^2\right). \tag{29}$$

The left-hand side of (29) is a measure of proximity of a DMRL distribution to an exponential distribution of the same mean μ, and the right-hand side of (29) is a function of this coefficient of variation.

Numerical studies given in Table 6 verify that our upper bound (28) beats the upper bound of Sengupta and Das [15] for DMRL distributions given by

$$P(X \geq a) \leq e^{-(a-\mu)/\mu}, \quad a \geq \mu, \tag{30}$$

for large values of a and for non-exponential DMRL distributions with $\left(\frac{\sigma}{\mu}\right) << 1$. However, if $\left(\frac{\sigma}{\mu}\right) \approx 1$, then the Sengupta & Das [15] upper bound (30) is better.

As another application, bounds for the mgf of k-out-of-n- system may be obtained from bounds on the mean residual life function of a k-out-of-n- system which are given in Raqab and Rychlik [13] in terms of bounds on component mean residual life functions.

We can also obtain bounds on tail probabilities of sums of DMRL random variables using just the means and variances of the random variables as done in From [8] for the NBUE case. In this paper, various upper bounds of Ahmad and Mugdadi [1] were utilized.

In real-life applications, the values of μ and σ^2 must be estimated by a sample mean and variance, respectively. Then, all bounds in this paper and all bounds of papers cited in this work are not guaranteed to be bounds on the moment generating function or on the tail probabilities in (28) when estimates of μ and σ^2 are used. However, this problem is mitigated in large samples by standard consistency results and laws of large numbers theorems. We are currently investigating bounds for moments and certain functions of moments of X for the DMRL class and hope to report on this in the future.

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